Auxiliary tables such as those of Borgen, Andoyer, Steinhauser, Gray, and Thoman, involving numbers to 10 or 11, 13, 20, 24, and 27 places, respectively, are investigated. The simplest method for a ten figure logarithm is believed to be the use of Steinhauser in connection with Schron's seven decimal table. A more extensive logarithm can easily be calculated with a slight increase in the arithmetic.

5. Professor Van Velzer's recollections were arranged to show four characteristics of Sylvester: physical peculiarities, versatility, modesty and intensity of interest.

6. In a popular form this paper discussed the hypothesis that the radiant energy of space is condensed in space into the atoms which are the constituent elements in the vast amounts of nebulosity that are recorded on our astronomical photographs; that this nebulosity is gathered in by the stars in their travels through space and that the energy which they contain is the source of the energies of the stars. The property of mass is acquired when the energy is locked up in the formation of the atom and is lost again when the atom passes out of existence in the interior of some star. This conception of the evolution and dissolution of matter leads to entirely new ideas as to the evolution of stars and galaxies.

7. Professor Rietz began with a discussion of the scope and limitations of the material in the third Carus Monograph which is an introduction to mathematical statistics. He described two general types of problems with which mathematical statistics deals. The first type of problem is concerned primarily with the description and characterization of a random sample drawn from a class or "population" of items. The second type of problem is concerned mainly with the question of making and testing the validity of inferences about the properties of the class or population from a knowledge of a random sample. After a sort of parallel presentation of certain fundamental concepts such as relative frequency relating to the sample and probability relating to the population, the paper gives a summary of the material in the monograph which deals with the following three topics which have been of dominant interest in recent progress in mathematical statistics: Generalization of frequency curves, correlation theory, and random sampling theory.

8. Two fundamental theorems concerning stereographic projection state that circles are transformed into circles, and that the transformation is conformal. The author proves these facts by using some theorems from projective geometry, keeping in mind the special metric properties associated with the absolute, i.e., the sphere circle at infinity. The proofs are interesting in that the projective theorems employed are of an elementary nature; for the most part they are the fundamental theorems concerning quadric surfaces and their reguli.

9. Some notes concerning algebraic geometry as developed by the Italians, and opportunities for study in Italy.

10. This paper treats three problems. First it exhibits a solution of the Diophantine equation $\sum 1/(x_1x_2 \cdots x_r) = 1$ whose left member consists of the sum of the reciprocals of the terms of the elementary symmetric function E_r of the n unknowns $x_1, x_2, \cdots, x_n$. This solution is then shown to contain the maximum x that can occur in a solution in positive integers of the equation. Furthermore this solution is the only one which contains the maximum x. The second problem treated is that of finding the maximum x (when $x_1 \leqq x_2 \leqq \ldots \leqq x_n$) that can occur in a solution in positive integers of the cyclo-symmetric equation

$$\frac{1}{x_1x_2} + \frac{1}{x_2x_3} + \cdots + \frac{1}{x_{n-1}x_n} + \frac{1}{x_nx_1} = 1 .$$

A unique result is obtained by the method which was used in handling the first problem. The third part of the paper exhibits an apparent maximum for Problem 2 in the list of problems proposed to readers of the Monthly by the author in his paper on "Diophantine Problems in Weighing."[1]

11. What effect will the 12 unit entrance requirements proposed by the North Central Association of Colleges and Secondary Schools have upon the present requirement of one unit in geometry and one unit in algebra? Will this plan result in such a lowering of standards that the number of students applying for admission without algebra or geometry will be increased? How shall such students be treated? Which of several suggested ways should be followed? (a) admit without two units of mathematics and require no mathematics in college; (b) admit without two units but require them to be made up without college credit, an opportunity to make up such work being (1) provided by the college, or (2) not provided by the college; (c) admit without two units, but opportunity given in college to make up such work with college credit.

BESSIE I. MILLER, *Secretary*

THE SIXTEENTH MEETING OF THE IOWA SECTION

The sixteenth regular meeting of the Iowa Section of the Mathematical Association of America was held in conjunction with the annual meeting of the Iowa Academy of Science at the University of Iowa, Iowa City, May 6 and 7, 1927.

The attendance was about forty including the following twenty-five members of the Association: R. P. Baker, A. H. Blue, E. W. Chittenden, Julia T.

[1] This Monthly, vol. 34 (1927), pp. 4–22.

Colpitts, N. B. Conkwright, Marian E. Daniells, Annie W. Fleming, D. Jackson, J. V. McKelvey, F. M. McGaw, C. A. Messick, E. E. Moots, E. A. Pattengill, J. F. Reilly, F. Reusser, H. L. Rietz, E. R. Smith, J. Theobald, J. S. Turner, L. E. Ward, C. W. Wester, A. E. White, R. Woods, C. C. Wylie.

The Chairman of the Section, J. V. McKelvey, presided at both the Friday afternoon and the Saturday morning sessions.

Dinner was enjoyed together Friday evening at the Iowa Memorial Union. At the business meeting which followed the program, the following were elected officers for 1927–1928: Chairman, ROSCOE WOODS, University of Iowa; Vice-chairman, E. E. MOOTS, Cornell College; Secretary-Treasurer, J. F. REILLY, University of Iowa.

The program consisted of nine papers and two addresses. The papers were as follows:

1. "Note on the hyperbola" by Professor ROSCOE WOODS, University of Iowa.

2. "Fitting of conic sections to intersecting straight lines," by Professor E. R. SMITH, Iowa State College.

3. "On folk algebra and problem solving," by Professor C. W. WESTER, Iowa State Teachers College.

4. "Thorndike's problem," by Professor WESTER.

5. "The expansion of an infinite product of polynomials into an infinite series," by Mr. C. A. MESSICK, University of Iowa.

6. "The meteor of January 2, 1927," by Professor C. C. WYLIE, University of Iowa.

7. "On the expected value of the product of certain statistical variables," by Professor H. L. RIETZ, University of Iowa.

8. "Asymptotic series," by Professor JULIA L. COLPITTS, Iowa State College.

9. "Note on the mean value theorem in the finite difference calculus," by Professor J. F. REILLY, University of Iowa.

Abstracts of these papers are given below, the numbers corresponding to those in the list of titles.

1. This note is the solution of a recent problem, No. 2409, proposed in Mathesis of December, 1926. It is proved that it is possible to draw four circles through the center of the hyperbola and a fixed point M on the curve and tangent to the hyperbola, the four points of contact being distinct from the point M. Further that the four points of contact are concyclic and that this circle passes through the center. The locus of this circle is found as M traces out the hyperbola.

2. This paper considers the properties of the family of conics which are tangent to two given intersecting lines at fixed points. The centers of the curves all lie on a straight line through the intersection of the given lines and

through the mid-point of the segment which joins the fixed points of tangency. The equation of the family, involving a parameter, may be readily obtained, and the values of the parameter for which a parabola, ellipses, hyperbolas, and a pair of straight lines are obtained may be determined. Geometric constructions are available for constructing the conic corresponding to any value of the parameter.

3. An algebra of positive numbers used more or less unconsciously by everybody solving arithmetic problems is here exhibited in the notation of ordinary algebra. Its chief principle is stated, and applications are given to problems to show its power and wide scope. It is offered as a substitute for the present confusion of methods for problem solving in arithmetic, being far simpler than ordinary algebra and incomparably more so than the current arithmetics.

4. The problem of rating computers where speed and accuracy are both taken into account has been a troublesome one. Various workers in the educational field have proposed tentative solutions, arrived at empirically. Thorndike proposed that it be solved by determining which of two or more computers would finish first a set of computations, if each computation must be checked and rechecked till two agreeing results are obtained. A mathematical solution of this problem is here given; it indicates that the only mark needed as a rating is the number right in a given time.

5. Let $\pi[1+V_n(x)]$ be absolutely convergent, and let $V_n(x)$ be a polynomial of fixed degree r. By use of the logarithmic and exponential series, expand the product into a power series $1+c_1x+c_2x^2+\cdots$. A formula is derived for each value of r which gives c_n in terms of the simple series $\sum a_n^{\alpha}b_n^{\beta}\cdots k_n^{\kappa}$ of the coefficients $a_n, \cdots k_n$, of V_n.

6. At about 6:02 P.M. January 2, 1927, a brilliant meteor passed over eastern Iowa. A total of considerably more than one hundred observations have been received by the University of Iowa. The observers at Iowa City considered it a little brighter than the full moon, while most of those at Dubuque, slightly farther away, considered it somewhat less bright. A computation of its path shows that it must have appeared about twenty-five miles north of Waterloo and at a height of about eighty miles, and crossed the Mississippi river just north of Burlington at a height of about twenty miles. The exact point of disappearance is not well fixed as we have insufficient data from Illinois. Our preliminary estimate for the velocity is about forty miles per second, which indicates that it came almost certainly from outside the solar system. It came from the direction of the constellation, Draco, near the point towards which the solar system is moving.

7. This paper by Professor Rietz deals with the expected value of the products of certain relative frequencies in which the expected value of the product of the variables is not equal to the product of their expected values.

8. In this expository paper Professor Colpitts discussed the history of asymptotic series, computation by means of them, and some of their properties.

9. In his paper Professor Reilly gave the analogues of Rolle's theorem and the mean value theorem for the finite difference calculus, and suggested that, if the mean value theorem be extended as in the infinitesimal calculus, Newton's formula of interpolation would result.

After the Friday evening dinner Professor J. V. McKelvey gave the retiring chairman's address on "Discontinuities and prerequisites," Professor E. R. Smith, Iowa State College, presiding. This was followed by a general and enthusiastic discussion.

At the Saturday morning session Professor Dunham Jackson, University of Minnesota, who was present by invitation, gave an address on "Trigonometric interpolation," Professor E. W. Chittenden, University of Iowa, presiding. This address is published in this number of Monthly.

Professor J. S. Turner, Iowa State College, exhibited a very old book on Trigonometry and Logarithms, "Mirifici logarithmorum canonis descriptio . . .," published at Edinburgh in 1614, and dedicated to the Prince of Wales (afterwards Charles I).

J. F. REILLY, *Secretary*

THE TWELFTH MEETING OF THE KANSAS SECTION

The twelfth regular meeting of the Kansas Section of the Mathematical Association of America was held at Topeka, February 5, 1927, in connection with the annual meeting of the Kansas Association of Mathematics Teachers. Two sessions were held. The first was a joint session with the Kansas Association. Professor W. H. Garrett, Chairman of the Kansas Section, presided part of the time during the forenoon (joint) session and all of the time during the afternoon session.

The attendance was fifty-six including the following twenty-seven members of the Association: C. H. Ashton, W. H. Andrews, Alice Austin, Wealthy Babcock, Florence Black, Lucy Dougherty, P. L. Evans, E. F. Farner, W. H. Garrett, R. W. Hart, Ina Holroyd, Emma Hyde, C. F. Lewis, W. H. Lyons, Anna Marm, Nina McLatchey, U. G. Mitchell, H. S. Myers, A. W. Philips, T. I. Porter, C. A. Reagan, Ethel Rumney, J. A. G. Shirk, G. W. Smith, Edith Steininger, W. T. Stratton, J. J. Wheeler.

The following officers were elected for the coming year: Chairman, U. G. MITCHELL, Vice-Chairman, H. S. MYERS, Secretary-Treasurer, LUCY DOUGHERTY.

The following papers by members of the Association were presented:

1. "The problem of the college freshman with no high school mathematics," by Professors J. A. G. SHIRK and H. S. MYERS.

2. "Relation of principals and teachers to the recent changes in high school mathematical requirements," by Principal E. F. FARNER.

3. "Certain tests for primality of integers," by Professor C. A. REAGAN.

4. "Some important points in the development of the theory of determinants," by Professor WEALTHY BABCOCK.

5. "The work of Euler, Laplace, and Lagrange relating to the stability of the solar system," by Professor PIUS PRETZ (representing an institutional member).

6. "An appreciation of the life and work of Sir Isaac Newton," by Professor G. W. SMITH.

Comments on the first two papers and abstracts of the others, numbered as in the above list of titles, follow:

1. and 2. These papers were presented at the joint session with the Kansas Association of Mathematics Teachers and were in the nature of discussions of the situation created in Kansas by a recent ruling of the State Board of Education permitting students to graduate from accredited high schools without any high school mathematics. Full summaries of these papers appeared in the April number of the Bulletin of the Kansas Association of Mathematics Teachers (Edited by Professor Ina E. Holroyd and published at Manhattan, Kansas).

3. Professor Reagan showed that all integers of the form $6n+1$ are primes except when n is of the form $6ar \pm (a+r)$. For a given integer of the form $6n+1$ the $6ar \pm (a+r)$ may be equated to the numerical value of n and a test for primality made by testing the resulting indeterminate equations for positive integral solutions. For certain rather large primes the labor involved in making such a test seemed not to be formidable. Similar forms for testing numbers of the form $6n-1$ were developed.

4. Professor Babcock's paper outlined the theory of determinants somewhat in the order of its historical development. Of the early work the contributions of only four men, Leibnitz, Cramer, Vandermonde and Cauchy, were considered. Cayley's expansion theorem, Jacobi's theorems, Sylvester's theorem, and theorems concerning the vanishing of a determinant and the independence of the minors of a determinant were cited as important theorems in the later development of the theory. A few special determinants, including symmetric determinants, Jacobians, and Wronskians, were defined and some theorems concerning them given. Brief mention was made of the theory of infinite determinants, developed since the publication of Hill's paper in 1877, and of the Fredholm's determinant and its application to integral equations.

5. Prior to the eighteenth century many theories had been advanced to account for the order, harmony, and stability of the solar system. Most notable among them were the *deus ex machina* of Newton, maintaining that a "powerful hand" intervened from time to time to set the celestial machinery right, and the "perpetual miracle" of Leibnitz. Professor Pretz attempted to show, in a non-technical way, that the work of Euler, Lagrange, and Laplace in mathematics, mechanics, and astronomy placed sufficient material in the hands of Laplace to prove that the order, harmony, and stability of the solar system were due to the operation of natural laws, the laws of gravitation.

6. Professor Smith pointed out that the two-hundredth anniversary of the death of Newton would occur on March 20, 1927, and that various mathematical societies were devoting meetings to a commemoration of his life and work. He then briefly recalled the chief events in Newton's life and his most important discoveries in mathematics and physics, and gave a short discussion of his law of universal gravitation in connection with recent discoveries and developments.

U. G. MITCHELL, *Secretary*

SOME NOTES ON TRIGONOMETRIC INTERPOLATION[1]

By DUNHAM JACKSON, University of Minnesota

The object of this paper is to call attention to certain relations involving the formulas of trigonometric interpolation with equidistant ordinates. Some of these relations, while they may not be new, are at any rate less familiar than the most fundamental facts in the theory of such interpolating formulas.

Let $f(x)$ be a given function of period 2π, for simplicity continuous. Let m be an arbitrary positive integer, and let $x_i = 2i\pi/m$, so that $x_0 = 0$, $x_m = 2\pi$, and the interval $(0, 2\pi)$ is divided into m equal parts by the points $x_1, \cdots, x_{m-1}$. If coefficients a_k, b_k are defined by means of the formulas

$$a_k = \frac{2}{m} \sum_{i=1}^{m} f(x_i) \cos kx_i, \qquad b_k = \frac{2}{m} \sum_{i=1}^{m} f(x_i) \sin kx_i,$$

it is well known that the expression

$$\begin{aligned} \phi_n(x) = {} & \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx \\ & + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx \end{aligned}$$

if m is odd, $m = 2n+1$, or

$$\begin{aligned} \psi_n(x) = {} & \tfrac{1}{2}a_0 + a_1 \cos x + \cdots + a_{n-1} \cos (n-1)x + \tfrac{1}{2}a_n \cos nx \\ & + b_1 \sin x + \cdots + b_{n-1} \sin (n-1)x \end{aligned}$$

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[1] Presented to the Iowa Section of the Association at Iowa City, May 7, 1927.

if m is even, $m=2n$, is a trigonometric sum of the nth order coinciding in value with $f(x)$ at each of the points x_i, and is the only sum of the nth order, or the only sum of the nth order lacking the term in sin nx, which has this property.

For comparison, let the Fourier coefficients of $f(x)$ be denoted by

$$\alpha_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx, \qquad \beta_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx.$$

If $\Delta x = 2\pi/m$, it is seen that

$$a_k = \frac{1}{\pi}\sum_{i=1}^{m} f(x_i)\cos kx_i\,\Delta x, \qquad \text{with a corresponding formula for } b_k.$$

Hence, if k is held fast, $\lim_{m=\infty} a_k = \alpha_k$, $\lim_{m=\infty} b_k = \beta_k$; the interpolating coefficients are approximations to the Fourier coefficients, to the extent that a sum of rectangles is an approximation to the area under a curve.

The formulas for the a's and b's, however, are deserving of closer examination. Apart from the factor $1/\pi$, each is of the form

$$\sum_{i=1}^{m} F(x_i)\Delta x, \tag{1}$$

where $F(x)$ is a function of period 2π. The trapezoid formula for approximating to the integral

$$\int_0^{2\pi} F(x)\,dx \tag{2}$$

would be

$$\Delta x\left[\tfrac{1}{2}F(x_0) + F(x_1) + \cdots + F(x_{m-1}) + \tfrac{1}{2}F(x_m)\right]. \tag{3}$$

But as $F(x_0) = F(x_m)$, by the hypothesis of periodicity, (3) is identical with (1). *Each interpolating coefficient is an approximation to the corresponding Fourier coefficient, not merely in the sense of the rectangle formula, but in the sense of the trapezoid formula of mechanical quadrature at the same time.*

The application of Simpson's rule to the integral (2), when m is even, yields

$$\tfrac{1}{3}\Delta x\left[F(x_0) + 4F(x_1) + 2F(x_2) + \cdots + 4F(x_{m-1}) + F(x_m)\right],$$

which is the same as

$$\tfrac{1}{3}\Delta x\left[4F(x_1) + 2F(x_2) + \cdots + 4F(x_{m-1}) + 2F(x_m)\right]. \tag{4}$$

This is not by any device of interpretation the same as (1). But let the corresponding quadrature formula be written down for the integral $\int_{x_1}^{x_1+2\pi} F(x)\,dx$, the value of which is the same as that of (2), since the integral of a periodic

function over a period interval is independent of the situation of the interval. The formula is

$$\tfrac{1}{3}\Delta x[4F(x_2) + 2F(x_3) + \cdots + 4F(x_m) + 2F(x_{m+1})],$$

which is the same as

$$\tfrac{1}{3}\Delta x[2F(x_1) + 4F(x_2) + 2F(x_3) + \cdots + 4F(x_m)], \tag{5}$$

since $F(x_{m+1}) = F(x_1)$. *The average of* (4) *and* (5) *reduces to* (1) *again; the interpolating coefficient is the average of the results obtained by two applications of Simpson's rule to the integral defining the Fourier coefficient*, and so represents an approximation at least as good, on general principles, as that to be expected from the use of Simpson's rule directly.

The same form of interpretation can be extended to higher formulas of mechanical quadrature, if m is divisible by the number of sub-intervals that are grouped together in setting up the quadrature formula, that is, by the degree of the polynomials from which the quadrature formula is obtained. And it is worthy of note that the values of m most likely to be used in practice, namely the number of hours in a day and the number of months in a year, are divisible by 2, 3, 4, 6, and 12.

Even if m were not divisible by 2, say, in connection with Simpson's rule, that rule could still be applied to the integral from 0 to 4π, which is equal to the double of the integral from 0 to 2π, because of the periodicity, and the interpretation of the interpolating coefficient as a result of quadratures by Simpson's rule would still be justified. For a_k, for example, may be regarded as defined by the formula

$$a_k = \frac{1}{2\pi}\int_0^{4\pi} f(x)\cos kx\,dx.$$

It must be admitted that attention has been directed to the excellence of the approximations, rather than to their shortcomings. It is apparent on renewed consideration that the approximations are unquestionably good only if m is large in comparison with k, and this is true only for the early terms in the interpolating sum of any specified order. Suppose for example that $f(x)$ is everywhere positive; then $f(x)\cos kx$ changes sign $2k$ times in a period, and if k is only a little less than $\frac{1}{2}m$ the quadrature formula uses only about one point for each change of sign, and promises, superficially at least, only a decidedly crude result.

This deficiency is particularly apparent on comparison of the formulas for the sum of the whole interpolating expression on the one hand and the partial sum of the Fourier series on the other. For simplicity let attention be restricted to the case that $m = 2n+1$. The formulas are respectively

$$\frac{1}{\pi}\sum_{i=1}^{2n+1} f(x_i)\frac{\sin(n+\frac{1}{2})(x_i-x)}{2\sin\frac{1}{2}(x_i-x)}\Delta x \quad \text{and} \quad \frac{1}{\pi}\int_0^{2\pi} f(t)\frac{\sin(n+\frac{1}{2})(t-x)}{2\sin\frac{1}{2}(t-x)}dt.$$

There is just one term in the sum for each arch of the sine in the numerator.

In the light of the last remarks, it is perhaps all the more noteworthy that the convergence properties of the interpolation formula and of the Fourier series correspond as closely as they do.[1] One is led to inquire, though with a rather definite expectation as to the result, whether the convergence properties would be preserved if the interpolating coefficients were replaced by quadrature formulas using a larger number of points, and so presumably approximating more closely to the actual values of the Fourier coefficients. A conclusion with regard to this problem may be formulated in the following

THEOREM. *For each positive integral value of p, let*

$$\phi_p(x) = \tfrac{1}{2}a_{0p} + a_{1p}\cos x + a_{2p}\cos 2x + \cdots + a_{pp}\cos px + b_{1p}\sin x + b_{2p}\sin 2x + \cdots + b_{pp}\sin px,$$

where

$$a_{kp} = \frac{2}{m_p}\sum_{i=1}^{m_p} f(x_i)\cos kx_i, \qquad b_{kp} = \frac{2}{m_p}\sum_{i=1}^{m_p} f(x_i)\sin kx_i,$$

$$x_i = 2i\pi/m_p,$$

the numbers m_p, for the successive values of p, being subject merely to the requirement that $m_p > 2p+1$. Then $\phi_p(x)$ will converge uniformly toward $f(x)$, as p becomes infinite, provided that $f(x)$ has a modulus of continuity $\omega(\delta)$ such that $\lim_{\delta=0}\omega(\delta)\log\delta = 0$, that is, if $f(x)$ satisfies one of the standard sufficient conditions for the convergence of the ordinary interpolation formula or the Fourier series.

The condition with regard to the modulus of continuity means that $[f(x_2)-f(x_1)]\log|x_2-x_1|$ approaches zero uniformly when x_2-x_1 approaches zero. A detailed presentation of the proof of the theorem is unnecessary for the technical worker in the field, and would be of little interest to other readers. Briefly, the proof is as follows.

If $f(x)$ were itself a trigonometric sum of order p, $\phi_p(x)$ would be identical with $f(x)$. For in any case $\phi_p(x)$ is merely the partial sum of the interpolating expression for $f(x)$ corresponding to a subdivision of the interval $(0, 2\pi)$ into m_p equal parts, through terms of order p; and when $f(x)$ is a trigonometric sum of order $p < \frac{1}{2}(m_p-1)$, the whole interpolating expression is identical with $f(x)$, and has no terms of order higher than p.

[1] Cf., e.g., Faber, *Über stetige Funktionen*, Mathematische Annalen, vol. 69 (1910), pp. 372–443; D. Jackson, *On approximation by trigonometric sums and polynomials*, Transactions of the American Mathematical Society, vol. 13 (1912), pp. 491–515; *On the accuracy of trigonometric interpolation*, the same Transactions, vol. 14 (1913), pp. 453–461.

Secondly, if $f(x)$ has M as an upper bound for its absolute value, but is otherwise unrestricted, $|\phi_p(x)| \leqq CM \log p$, where C is a constant independent of p, m_p, and the form of the function $f(x)$. For ϕ_p can be written by means of a well-known identity in the form

$$\phi_p(x) = \frac{1}{m_p} \sum_{i=1}^{m_p} f(x_i) \frac{\sin (p + \frac{1}{2})(x_i - x)}{\sin \frac{1}{2}(x_i - x)}.$$

The number of terms in this sum for which $\frac{1}{2}(x_i - x)$ differs from 0 or from π by not more than $1/p$ is within one unit of the number of times that an interval of length π/m_p is contained in an interval of length $2/p$, and does not exceed a constant times m_p/p; the absolute value of the quotient of sines can not be more than $2p+1$; and so the sum of this group of terms, with the factor $1/m_p$, does not exceed a constant multiple of M. In the remaining terms it is to be noted that $|\sin (p+\frac{1}{2})(x_i - x)| \leqq 1$, and that $\sin \frac{1}{2}(x_i - x)$ is not less in absolute value than $2/\pi$ times the difference between $\frac{1}{2}(x_i - x)$ and the nearer of the numbers 0, π, since $\sin y/y \geqq 2/\pi$ for $0 < y \leqq \pi/2$, and $\sin y/(\pi - y) = \sin (\pi - y)/(\pi - y) \geqq 2/\pi$ for $\pi/2 \leqq y < \pi$. Hence the sum of these terms, with the factor $1/m_p$ included, is not greater than a constant multiple of M/m_p times the sum of the reciprocals of a set of numbers succeeding each other at intervals of π/m_p in the range from $1/p$ to $\pi/2$. And this sum of reciprocals lacks only the factor π/m_p to make it an approximation to the value of the integral $\int dt/t$, extended from $1/p$ to $\pi/2$, and so does not exceed a constant multiple of $m_p \log p$. Thus the assertion at the beginning of the paragraph is justified. It follows immediately that

$$|f(x) - \phi_p(x)| \leqq M(C \log p + 1).$$

Thirdly, suppose $f(x)$ can be approximately represented by a trigonometric sum $T_p(x)$ of order p so that $|f(x) - T_p(x)|$ does not exceed ϵ_p. The difference between $f(x)$ and $\phi_p(x)$ is the sum of the corresponding differences formed for $f(x) - T_p(x)$ and for $T_p(x)$ respectively. The first of these differences does not exceed $\epsilon_p(C \log p + 1)$, by the preceding paragraph, and the second is identically zero, by the paragraph before that. Hence

$$|f(x) - \phi_p(x)| \leqq \epsilon_p(C \log p + 1).$$

Finally, if $f(x)$ has a modulus of continuity $\omega(\delta)$ such that $\lim_{\delta=0} \omega(\delta) \log \delta = 0$, it is known[1] that trigonometric sums $T_p(x)$ can be constructed so that

$$\lim_{p=\infty} \epsilon_p \log p = 0.$$

From this follows the desired conclusion that $\phi_p(x)$ converges uniformly toward $f(x)$.

[1] Lebesgue, *Sur les intégrales singulières*, Annales de la Faculté de Toulouse, (3), vol. 1 (1910), pp. 25–117; pp. 115–116; D. Jackson, *On the approximate representation of an indefinite integral . . . .*, Transactions of the American Mathematical Society, vol. 14 (1913), pp. 343–364; p. 350.

FUNCTIONS OF CLOSEST APPROXIMATION ON AN INFINITE RANGE[1]

By W. D. CAIRNS, Oberlin College

The purpose of this paper is to extend to the interval $(-\infty, \infty)$ a theorem of Dunham Jackson[2] on the existence of the polynomial of specified degree, or of a linear combination of n given functions, which gives the best approximation to a given continuous function in a given interval (a, b). This extension is accomplished by the expedient of introducing the factor $e^{-x^2/4}$. The general argument is quite similar to that of the paper referred to; the analogy is shown more fully by the rather free use of this exponential factor.

Let $p_1(x), p_2(x), \cdots, p_n(x)$ be n functions, continuous for finite x, and such that one can find a positive constant h so that

$$|p_i(x)| \leqq |x|^h \qquad (i = 1, 2, \cdots, n) \text{ for } |x| > 1.$$

Let $$\phi(x)/e^{-x^2/4} = c_1 p(x) e^{-x^2/4} + c_2 p_2(x) e^{-x^2/4} + \cdots + c_n p_n(x) e^{-x^2/4}$$

be an arbitrary linear combination of these functions with constant coefficients and multiplied by $e^{-x^2/4}$. It will then, by hypothesis, be true that

$$\left|\frac{\phi(x)}{e^{-x^2/4}}\right| \leqq C|x|^h e^{-x^2/4} \qquad C = |c_1| + \cdots + |c_n| \text{ for } |x| > 1.$$

LEMMA I. *There exists a constant K, completely determined by the system of functions $p_i(x)$, such that $|c_i| \leqq K$ for all functions $\phi(x)$, $i = 1, 2, \cdots, n$.*

For each value of k let the coefficients in

$$\Phi_k(x)/e^{-x^2/4} = c_{1k} p_1(x) e^{-x^2/4} + \cdots + c_{nk} p_n(x) e^{-x^2/4}$$

be determined so that

$$\int_{-\infty}^{\infty} p_i(x)\Phi_k(x)\,dx = 0, \qquad i \neq k\,; \qquad \int_{-\infty}^{\infty} p_k(x)\Phi_k(x)\,dx = 1.$$

This is possible because of the fact that the functions $p_i(x)$ are linearly independent, just as in Professor Jackson's proof. Since for every $\Phi_k(x)$ it is true, as in the case of $\phi(x)$, that

$$\left|\frac{\Phi_k(x)}{e^{-x^2/4}}\right| \leqq C_k|x|^h e^{-x^2/4}, \qquad C_k = |c_{1k}| + \cdots + |c_{nk}|,$$

it follows that

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[1] Read before the Southern California Section of the Mathematical Association of America, Los Angeles, Calif. March 12, 1927.

[2] Transactions of the American Mathematical Society, vol. 22 (1921), pages 117–128.

$$\left| \int_{-\infty}^{\infty} \frac{\phi(x)}{e^{-x^2/4}} \frac{\Phi_k(x)}{e^{-x^2/4}} dx \right| \leqq CC_k \int_{-\infty}^{\infty} x^{2k} e^{-x^2/2} dx = K,$$

a finite constant, it being easy to show that the last (improper) integral fulfills the test for convergence. But by definition of $\Phi_k(x)$ it is also true that

$$\int_{-\infty}^{\infty} \frac{\phi(x)}{e^{-x^2/4}} \frac{\Phi_k(x)}{e^{-x^2/4}} dx = \int_{-\infty}^{\infty} \sum_{i=1}^{n} c_i p_i(x) e^{-x^2/4} \cdot \frac{\Phi_k(x)}{e^{-x^2/4}} dx = c_k \int_{-\infty}^{\infty} p_k(x) \Phi_k(x) dx = c_k.$$

Hence the c_k are bounded in absolute value by K, this constant depending on the coefficients in $\phi(x)$ inasmuch as the constant factor in the upper bound for $|\phi(x)|e^{x^2/4}$ depends on them.

Let m be a positive constant greater than 1. Set

$$\Delta_m = \int_{-\infty}^{\infty} e^{-x^2/2} \left| \frac{\phi(x)}{e^{-x^2/4}} \right|^m dx; \qquad \Delta = \int_{-\infty}^{\infty} e^{-x^2/2} dx = (2\pi)^{1/2}.$$

LEMMA II. *There exists a positive constant K_1, completely determined by the system of functions $p_i(x)$ and independent of m, such that*

$$|c_i| \leqq K_1(\Delta + \Delta_m), \qquad (i = 1, 2, \cdots, n) \text{ for all functions } \phi(x).$$

As in Professor Jackson's paper, since $m > 1$,

$$|\phi(x)/e^{-x^2/4}| \leqq |\phi(x)/e^{-x^2/4}|^m$$

unless the left member of the inequality is <1; hence in any case

$$|\phi(x)/e^{-x^2/4}| \leqq 1 + |\phi(x)/e^{-x^2/4}|^m$$

for any fixed $m > 1$, and for all values of x. Hence

$$\int_{-\infty}^{\infty} e^{-x^2/2} |\phi(x)/e^{-x^2/4}| dx \leqq \Delta + \Delta_m,$$

and for any finite value of x

$$\left| \int_{-\infty}^{x} e^{-x^2/2} [\phi(x)/e^{-x^2/4}] dx \right| \leqq \int_{-\infty}^{\infty} e^{-x^2/2} |\phi(x)/e^{-x^2/4}| dx \leqq \Delta + \Delta_m.$$

The n functions $\int_{-\infty}^{x} e^{-x^2/2} p_i(x) e^{-x^2/4} dx \qquad (i = 1, 2, \cdots, n)$

are linearly independent, since a linear relation between them would give by differentiation a linear relation connecting the functions $p_i(x)$. And since the last inequality may be written

$$\left| c_1 \int_{-\infty}^{x} e^{-x^2/2} p_1(x) e^{-x^2/4} dx + \cdots + c_n \int_{-\infty}^{x} e^{-x^2/2} p_n(x) e^{-x^2/4} dx \right| \leqq \Delta + \Delta_m,$$

the hypothesis of Lemma I is fulfilled, and by the analysis used in the latter part of the proof of Lemma I,

$$|c_i| \leqq C_k \int_{-\infty}^{\infty} (\Delta + \Delta_m) |x|^k e^{-x^2/4} dx \leqq K_1(\Delta + \Delta_m).$$

Let $f(x)$ be a function continuous for finite x, $f(x)/e^{-x^2/4}$ vanishing at $\pm\infty$ like $|x|^{h'}e^{-x^2/4}$; i.e., when $|x|>1$,

$$|f(x)/e^{-x^2/4}| \leqq |x|^{h'}e^{-x^2/4} \qquad (h' \text{ a positive constant}).$$

Also let

$$\Delta' = \int_{-\infty}^{\infty} e^{-x^2/2} |x|^{h'} e^{-x^2/4} dx\,; \qquad \delta_m = \int_{-\infty}^{\infty} e^{-x^2/2} |\{f(x) - \phi(x)\}/e^{-x^2/4}|^m dx.$$

LEMMA III. *For all functions* $\phi(x)$,

$$|c_i| \leqq K_1(\Delta' + \Delta + \delta_m) \qquad (i = 1, 2, \cdots, n),$$

where K_1 *is the positive constant of Lemma II.*

Since, as in the last argument,

$$|\{f(x) - \phi(x)\}/e^{-x^2/4}| \leqq 1 + |\{f(x) - \phi(x)\}/e^{-x^2/4}|^m,$$

we have for every $\phi(x)$

$$|\phi(x)/e^{-x^2/4}| \leqq |x|^{h'}e^{-x^2/4} + 1 + |\{f(x) - \phi(x)\}/e^{-x^2/4}|^m\,;$$

whence

$$\int_{-\infty}^{\infty} e^{-x^2/2} |\phi(x)/e^{-x^2/4}| dx \leqq \Delta' + \Delta + \delta_m.$$

Hence, by the argument in the last part of Lemma II, $|c_i| \leqq K_1(\Delta' + \Delta + \delta_m)$.

Existence of an approximating function. If the function $f(x)$, the system $p_i(x)$, and the exponent m are given, and the coefficients c_i are regarded as undetermined, the value of δ_m, which is a function of the c_i's, has a lower limit γ_m which is necessarily positive or zero. Since for a given function $f(x)$ there is an infinite number of ways of choosing the coefficients $c_i^{(j)}$, $j = 1, 2, \cdots$, and these by Lemma III are in a bounded region in an n space (regarding these as the coordinates of a point for each value of j), since

$$|c_i^{(j)}| \leqq K_1(\Delta' + \Delta + \delta_m),$$

there will be a set of limiting values for these which will yield a $\phi(x)$ for which δ_m attains its lower limit. In analogy with Professor Jackson's work, this may be called an approximating function for the exponent m.

REMARK: For $m=2$, this means that there exists a set of constants c_i for which

$$\delta_m = \int_{-\infty}^{\infty} [f(x) - \phi(x)]^2 dx$$

has a minimum, actually attained.

The writer is indebted to Professor Jackson for suggestions on details of the proof as well as for a communication on a briefer proof of the final theorem, when the squares of $f(x)$ and $p_i(x)$ are integrable from $-\infty$ to ∞, covering at least the case $m=2$.

THE ELDER ĀRYABHAṬA AND THE MODERN ARITHMETICAL NOTATION

By SĀRADĀKĀNTA GĀṄGULI, Ravenshaw College, Cuttack, India

Kaye writes[1] that the system of numeral notation by words, which is still popular in India, was introduced into India in the ninth century A.D. But any investigator who takes the trouble of reading the *original* works of Varāhamihira and his successors is sure to be convinced of the incorrectness of this statement. Burgess, Burnell, and Bühler are of the opinion that the system presupposes the existence of the modern arithmetical notation. On the authority of Barth, Burnell admits that the modern arithmetical notation was known to Āryabhaṭa. He also asks:[2] "May not Āryabhaṭa be the discoverer of the decimal notation in India?" In this paper it is proposed to give an answer to this question.

In India there lived two mathematicians and astronomers bearing the name Āryabhaṭa.[3] The elder Āryabhaṭa was born in 476 A.D. and is known as Āryabhaṭa of Kusumapura (modern Pāṭnā). He devised an alphabetic scheme of expressing numbers, which, although certainly based on the modern notation, does not recognise the value of position. The younger Āryabhaṭa lived in the tenth century A.D. and is the author of another alphabetic scheme which has been termed the *kaṭapayādi* system.[4] This latter scheme is nothing but the modern arithmetical notation with the signs replaced by letters. It is Āryabhaṭa of Kusumapura to whom Burnell refers.

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[1] *Indian Mathematics* (Calcutta, 1915), p. 31.

[2] *South Indian Palæography*, p. 62, foot-note 4.

[3] For the controversy on the two Āryabhaṭas the reader is referred to B. B. Datta's article on the subject published in the Bulletin of the Calcutta Mathematical Society, Vol. 17, pp. 59–74.

[4] Burnell and Bühler have described the later *kaṭapayādi* system and not the one devised by the younger Āryabhaṭa.

On an examination of the elder Āryabhaṭa's rules for the extraction of square and cube roots, Rodet writes:[1] "Donc, non seulement Āryabhaṭa opérait sur des nombres écrits *en chiffres avec valeur de position et zéro*, mais la pratique de ces sortes d'opérations était déjà familière à l'époque où il écrivait, ce qui suppose qu'elle existait déjà depuis un certain temps." But Kaye holds a diametrically opposite view and remarks that "there is not in any part of Āryabhaṭa's work the remotest indication of a notation with 'place values'."[2] Kaye's writings have modified the opinion of foreign scholars who now accept the 9th century A.D. as the earliest period when the modern notation was in use in India. I have elsewhere[3] shown that the above remark of Kaye is not supported by Āryabhaṭa's rules for the extraction of square and cube roots which Kaye has not been able to interpret correctly. If we assume, for the sake of argument only, that the words *vargāt, avargāt, ghanāt, aghanāt* which occur in the rules refer, as has been supposed by Kaye, not to places but to parts of the number whose root is to be found, the classification of the elements of a number into square (*varga*) and non-square (*avarga*) or into cubic (*ghana*) and non-cubic (*aghana*) is intelligible; but we fail to understand how the elements of a number can be classified as *cubic, second non-cubic*, and *first non-cubic*. Again, why should the *second non-cubic part* be taken before the *first* in the extraction of the cube root?[4]

Āryabhaṭa's rules for the extraction of square and cube roots indirectly prove that he was acquainted with the modern arithmetical notation. Direct proof, too, of this point is not lacking. The elder Āryabhaṭa has devoted the second chapter of his work, known as the *Āryabhaṭīyaṃ*, to mathematics. The second verse of this chapter contains an enunciation of the modern arithmetical notation. The verse runs thus:

Ekaṃ ca[5] daśa ca śatañca
sahasramayutaniyute tathā prayutam
Koṭyarbudañca vṛndam
sthānat sthānaṃ daśaguṇaṃ syāt.

Rodet translates this verse as follows:

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[1] Journal Asiatique, May and June, 1879, p. 408.

[2] Journal of the Asiatic Society of Bengal, July, 1907, p. 494.

[3] Journal of the Bihar & Orissa Research Society, March, 1926, pp. 78–82.

[4] Kaye's translation of Āryabhaṭa's rule for the extraction of the cube root is as follows:—"Multiply the square of the root of the cubic quantity by three and divide the second non-cubic part by the product. Multiply the square of this by three times the preceding and subtract the product from the first non-cubic. Then the cube is to be subtracted from the cube." (Journal of the Asiatic Society of Bengal, March, 1908, p. 120).

[5] I have introduced this word of a single syllable for the sake of the metre. The reading 'ekaṃ daśātha tu śataṃ sahasram etc.' suggested in Kern's edition of the *Āryabhaṭīyam*, page 18, footnote, involves more changes.

"*Eka, daçan, çata, sahasra, ayuta, niyuta, prayuta, koṭi, arbuda, vṛnda*, sont, de place en place, décuples l'un de l'autre" (Journal Asiatique, May & June, 1879, p. 397).

The words 'de place en place' suggest that Rodet takes *eka, daśa, śata*, &c., as the names of places, in which case he makes a near approach to the real meaning of the verse. This suggestion seems to be confirmed by his correct translation[1] of the corresponding verses of Bhāskara which he introduces by way of explanation. Here he says that *eka, daśa, śata*, &c., are the successive places ("les PLACES successives"). But the very significant omission on his part to urge this verse in support of his view that Āryabhaṭa was acquainted with the modern notation shows that he has failed to hit upon the right meaning of the verse and taken *eka, daśa, śata*, &c., as numerical units. As Rodet thinks that the modern notation existed in India even before the time of Āryabhaṭa, the introduction of the words 'de place en place' after a number of numerical units does not lead to any inconsistency as it does in the case of those who hold the opposite view. But his translation, we are bound to observe, is not correct. The words *sthānāt sthānaṃ* do not mean 'from place to place' (de place en place). The verb *syāt* (lit., 'should be') is in the singular number and has for its nominative *sthānaṃ* (place) and not *eka, daśa*, &c.; and the adjective *daśaguṇaṃ* is the subjective complement to the verb *syāt*.

Kaye translates the verse as follows:

"Units, tens, hundreds, thousands, tens of thousands, hundreds of thousands, millions, tens of millions, hundreds of millions, thousands of millions. In these each succeeding place is ten times the preceding place." (Journal of the Asiatic Society of Bengal, March, 1908, p. 117).

This translation consists of two parts. The first part simply names ten numerical units. The second part states the relation between the values of any two consecutive places. Is there any difference between the second part of the above translation and the principle of place-value which is the basis of our modern notation? Certainly not. Yet Kaye sticks to his opinion that the principle of place-value was unknown to Āryabhaṭa. To establish a connection between the two parts Kaye has, according to the usual custom which is to be followed in explaining Sanskrit verses, introduced the words "In these" which mean 'in the numerical units just mentioned'. But, how can one find *places* in those numerical units unless places have been assigned to them?

This defect in Kaye's translation seems to have attracted Fleet's notice. Accordingly, Fleet, who agrees with Kaye so far as the first part is concerned, translates[2] the second part, viz., *sthānāt sthānaṃ daśaguṇaṃ syāt*, as "from

[1] Journal Asiatique, May and June, 1879, p. 404.

[2] Journal of the Royal Asiatic Society, 1911, p. 116.

place to place each is a multiple by ten." Hence, Fleet's interpretation of the verse is practically the same as that given by Rodet. The remarks which have been made on the latter are, therefore, equally applicable to the former.

That none of the above three translations gives Āryabhaṭa's meaning of the verse follows from another consideration also. Āryabhaṭa has omitted the first four rules. It is, therefore, very unlikely that he should give the names of the numerical units which were too well-known in his time.

Let us now try to find the true interpretation of the verse. Let us first take the expression *sthānāt sthānaṃ daśaguṇaṃ syāt*. Here the verb *syāt* is the *vidhiliṅ* form of the verbal root *as* (to be). Hence it implies a *vidhi* or rule and should, therefore, be translated as 'should be, as a rule.' The translation of the expression should, therefore, be: 'each place should, as a rule, be ten times the next (lower) place.' The remaining part of the verse fits in with this expression only when *eka*, *daśa*, &c., are taken for the names of the places (*sthānānāṃ saṃjñāḥ*) after Bhāskara. Mahāvīra, who lived nearly three centuries and a half after the elder Āryabhaṭa, puts the same interpretation on the words *eka*, *daśa*, *śata*, &c. He writes:

Ekaṃ tu prathamaṃ sthānaṃ dvitīyaṃ daśasaṃjñikaṃ,
Tṛtīyaṃ śatamityāhuḥ caturthaṃ tu sahasrakaṃ,
etc. etc. etc.

(*Gaṇita-sāra-saṃgrahaḥ*, Rangācārya's edition, text, page 7).

Eka is the first place; the second (place) has *daśa* for its name (saṃjñā); (they) call the third (place) as *śata* and the fourth (place) as *sahasraka* (i.e. *sahasra*), etc.

This interpretation of the words *eka*, *daśa*, *śata*, &c. has passed on from Mahāvīra to Bhāskara through generations of teachers and must have been handed down to Mahāvīra in the same way. It should be used also in understanding Āryabhaṭa's verse under consideration. The meaning of the verse, therefore, seems at first sight to be as follows:

Eka, daśa, śata, sahasra, ayuta, niyuta, prayuta, koṭi, arbuda, and vṛnda are the names of the first ten places of which each place is ten times the next lower place.

This interpretation, like those given by Rodet, Kaye, and Fleet, is open to the following objections:

(a) Like the other Indian *sūtrakāras* (or framers of rules or formulæ) Āryabhaṭa is always brief to a fault. He is never guilty of a redundant statement. And it goes without saying that, whether *eka*, *daśa*, *śata*, &c., stand for numerical units or for places, the terms beginning with *daśa* are each ten times the next lower one. Is it then likely that Āryabhaṭa should, for once only, be so verbose, which is contrary to his nature as a *sūtrakāra*,and should add, by way of explanation, a statement which is only too obvious?

(b) According to the above interpretations *eka* is ten times the next lower place or unit. This would go to prove that Āryabhaṭa contemplated decimal fractions also. But there is no evidence in support of this conclusion.

(c) No one can deny that Āryabhaṭa speaks of at least nineteen places in the verse giving his alphabetic scheme.[1] Why, then, does he apply the rule 'each place is ten times the next lower place' to ten places or numerical units only? Is not the rule applicable to higher places or numerical units?

None of the above interpretations is, therefore, correct. I would, therefore, state the meaning of the verse thus:

The names of the first ten places are *eka*, *daśa*, *śata*, *sahasra*, *ayuta*, *niyuta prayuta*, *koṭi*, *arbuda*, *vṛnda;* the names of the higher places are to be found by applying the rule, viz., 'each place should be ten times the next lower place,' and by remembering that each place is named after the unit to which it has been assigned, as shown by the first ten names.

Āryabhaṭa names the first ten places only, probably because they are quite sufficient for all practical purposes.[2] It may be noted here that the names of decimal units higher than *vṛnda* have existed in India from before the time of Āryabhaṭa so that his verse under consideration enables us to find the names of as many places as there are decimal units with special names.

It will thus be seen that Āryabhaṭa's verse quoted above contains an exposition of a place-value decimal notation. It has been shown elsewhere[3] that Āryabhaṭa's rule for the extraction of the cube root necessitates the left-to-right order of arrangement of the component parts of a number in descending order of magnitude. We, therefore, conclude that the modern arithmetical notation was given by Āryabhaṭa in the above verse. This conclusion follows also from a comparison of the verse with the corresponding verses of Mahāvīra and Bhāskara who were undoubtedly too familiar with the modern notation.

We have assumed that the elder Āryabhaṭa is the author of the above verse. And we are justified in doing so. Both the Āryabhaṭas have composed their works in the *Āryā* metre and used its various forms. Hence an examination

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[1] The second half of the verse giving Āryabhaṭa's alphabetic scheme is as follows:—

'Khadvinavake svarā nava varge-avarge navāntyavarge vā.' Here 'khadvinavake' means 'in 18 places.' Fleet takes 'navāntyavarge' to mean 'in the nineteenth place.' But how can 9 vowels be used in the nineteenth place while Āryabhaṭa assigns each vowel to two places? Hence Fleet's interpretation seems to be wrong. The expression 'navantyavarge' is formed of the two words 'nava' and 'antyavarge.' Here 'antyavarge' can only mean 'in the following group (of 18 places)' so that the number of places contemplated by Āryabhaṭa knows no limit (See Bulletin of the Calcutta Mathematical Society, Vol. 17 (1926), pp. 195–196).

[2] Fleet thinks the reason to be that "none of the practical and fundamental elements, which it was absolutely necessary to state in the *Dasagītikāsūtra* (i. e. Chapter I of the *Āryabhaṭīyam*) runs beyond ten places of figures." (Journal of the Royal Asiatic Society, 1911, p. 116).

[3] Journal of the Bihar and Orissa Research Society, March, 1926, pp. 78–82

of the metre will not help us. Let us first consider the style. Leaving out of consideration those verses which contain numbers expressed in alphabetic notations, the style of the elder Āryabhaṭa is more lucid and graceful than that of the younger. And the style of the verse under consideration differs from that of the author of the *kaṭapayādi* system and is quite in keeping with the style of the astronomer who gives 476 A.D. as the date of his birth. Secondly, if the younger Āryabhaṭa had been the author of the above verse, its place would have been between the first and the second verse of the first chapter as given in Sudhākara Dvivedī's edition (Benares, 1910) of his work *Mahāsiddhānta*. For, the second verse gives the *kaṭapayādi* system which, as stated before, is the modern notation with the signs replaced by consonants. Unlike the *Āryabhaṭīyaṃ* the *Mahāsiddhānta* abounds in numbers having more than ten places of figures.[1] Why, then, should its author give the names of ten places only? The reason cannot be brevity, as he has not been brief in other cases. The rule for the extraction of the cube root has been stated by him in two verses and a half whereas his senior namesake has compressed it in a single similar verse. Again, as the modern notation and the names of places were too well-known when the younger Āryabhaṭa wrote, it would have been quite sufficient to name a few places only if the consideration of brevity induced him not to mention the names of places frequently used in his work. For example, brevity and clarity could be simultaneously secured by formulating the rule in the *Upagītikā* formt of he *Āryā* metre as follows:

Ekadaśaśatādināṃ sthānāt sthānaṃ
daśaguṇaṃ syāt

One might ask, 'Why did not the elder Āryabhaṭa simplify the verse in this way?' Our answer is that he was the first to record the principle of place-value which does not seem to have gained sufficient popularity when the *Āryabhaṭīyaṃ* was composed. Thirdly, a comparison of the names of places contained in the verse under consideration with those given by Mahāvīra (850 c.), Śrīdhara (1020 c.), and Bhāskara (1150 c.) lends additional support to our conclusion as to the authorship of the verse. The fact that the above three mathematicians working respectively in Mysore, Bengal, and Rājputanā—provinces of India widely separated from one another—in different centuries use the term *lakṣa* instead of the term *niyuta*, points strongly to the great extent of popularity which the term *lakṣa* gained throughout India in those days and which it still enjoys. If the younger Āryabhaṭa (10th century) had been the author of the

[1] For 11 places of figures see I, 7, 8; II, 5, 6; XIV, 19, 33, 41.
For 12 places of figures see II, 17; XIV, 25, 26, 30, 32; XVII, 8, 13, 15, 27.
For 13 places of figures see I, 13, XIV, 20, 21, 22, 23, 24, 34, 35, 36, 37, 39.
For 16 places of figures see I, 35.
For 17 places of figures see I, 32.

verse, he would have, in all probability, replaced the term *niyuta* by *lakṣa* like his predecessor Mahāvīra and his successors Śrīdhara and Bhāskara.

We are now in a position to return to Burnell's query quoted above. Both Āryabhaṭa and Brahmagupta have omitted the easier topics from the chapters of their works dealing with mathematics. But while the enunciation of the modern arithmetical notation finds a place in the elder Āryabhaṭa's work, Brahmagupta considers it too common-place for inclusion in his work. Āryabhaṭa would have certainly omitted it on account of its simplicity if it were popular when the *Āryabhaṭīyaṃ* was written or if it had occurred in a previous work. This leads us to conclude that the *Āryabhaṭīyaṃ* is the first work in which the modern arithmetical notation has been explained. So we are inclined to answer Burnell's query in the affirmative.

THE EVOLUTES OF A CERTAIN TYPE OF SYMMETRIC PLANE CURVES

By J. B. REYNOLDS, Lehigh University

The following ten theorems have been proved regarding the parabola and its evolutes.[1]

I. All evolutes of the parabola are symmetrical with respect to the axis of the parabola.

II. Every evolute of the parabola of odd order has one cusp only and it lies upon the axis of the parabola. Evolutes of even order have no cusps.

III. The curvature of the evolutes of the parabola at corresponding points decreases as the order of the evolutes increases.

IV. The curvature of each evolute is greatest where the evolute crosses the axis of the parabola.

V. Each evolute of the parabola of odd order has a point in common at its cusp with the evolute of next higher order at its vertex.

VI. The cusps on all evolutes of the parabola of order $4n+1$ lie within the parabola and those of order $4n-1$ lie without the parabola.

VII. The vertices of all evolutes of the parabola of order $4n+2$ lie within the parabola and those of order $4n$ lie without the parabola.

VIII. The vertices of the successive evolutes lie at increasingly greater distances from the vertex of the parabola.

IX. All evolutes of the parabola of odd order are concave towards the tangent at the vertex of the parabola.

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[1] See J. B. Reynolds, *Evolutes of the Parabola*. Proceedings of Pennsylvania Academy of Science vol. 1.

X. Evolutes of the parabola of even order beginning with the second have concavity alternately opposite to and in agreement with that of the parabola.

The question arises as to how many of these or similar theorems may be proved to hold for a general type of real curves of which the parabola is a particular case. The parabola and its evolutes of even order are found to have many common properties and may be said to belong to type C as defined below. The evolutes of odd order are of a different sort and do not fall under the definition of type C curves. A close study has led to defining the curves under consideration as those having the following characteristics:

(a) They are plane analytic symmetric curves.

(b) They have continuous evolutes.

(c) Each one consists of one real branch extending to infinity.

(d) They have no real point singularities.

Such a curve is here called a curve of type C. The definition of type C curves is designed to be such that a number of important theorems may be proved for the curve of this type and its evolutes. Furthermore, every evolute of even order satisfies the same conditions. Examples of type C curves are the parabola, its evolutes of even order, one branch of the hyperbola or one branch of $x=\sec y$.

The characteristic (b) might seem unfamiliar and hard to test for elementary curves but may, in fact, be readily applied as will become evident presently. Several of the theorems have to do, also, with the evolutes of a curve of type C.

Any plane curve $y=f(x)$, except the straight line, for which dy/dx is defined, may be represented by the vector equation $r=F(\theta)i+G(\theta)j$ in which i and j are unit vectors in a direction parallel respectively to a pair of mutually perpendicular axes and θ is the angle which the tangent to the curve makes with a fixed line of the plane.[1] For example, let i and j be parallel to the x- and y-axes respectively and the fixed line be the x-axis. This representation then corresponds to the parametric equations $x=F(\theta)$, $y=G(\theta)$, in which $\tan\theta=dy/dx$. To illustrate: For the parabola $y^2=4ax$, taking as parametric equations $x=at^2$ and $y=2at$, we find $dy/dx=\tan\theta=1/t$, whence $t=\cot\theta$ and the corresponding vector equation is $r=ai\cot^2\theta+2aj\cot\theta$.

THEOREM I. *The necessary and sufficient condition for the plane curve $r=F(\theta)i+G(\theta)j$ and all its evolutes to have finite coordinates corresponding to every finite point on r is that $F, F', \cdots, F^{(n)}$ and $G, G', \cdots, G^{(n)}$ shall be finite, the superscripts referring to derivatives with respect to the angle θ which the tangent to the curve makes with a fixed line of the plane.*

For, if $r_1=F_1i+G_1j$, $r_2=F_2i+G_2j$, $\cdots$, $r_n=F_ni+G_nj$ represent the first n evolutes of r we have[2]

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[1] See, for example, W. Blaschke, *Vorlesungen über Differential-geometrie* I.

[2] See Scheffer's *Anwendung der Differential und integral Rechnung auf Geometrie* vol. 1. page 68.

$$F_1 = F - G' \qquad G_1 = G + F'$$
$$F_2 = F - 2G' - F'' \qquad G_2 = G + 2F' - G''$$
$$F_3 = F - 3G' - 3F'' + G''' \qquad G_3 = G + 3F' - 3G'' - F'''$$
$$\cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \qquad \cdot \quad \cdot \quad \cdot$$
$$F_{n+1} = F_n - G_n' \qquad G_{n+1} = G_n + F_n'.$$

It is readily seen that $F_1, F_2, \cdots, F_n$ and $G_1, G_2, \cdots, G_n$ will be finite if $F, F', \cdots, F^{(n)}$ and $G, G', \cdots, G^{(n)}$ are finite, and further, if any of the latter be infinite, at least one of the former will be infinite. The condition stated in the theorem is therefore necessary and sufficient.[1]

THEOREM II. *If for every point on a curve there corresponds a finite point on each of its evolutes, neither the curve nor any of its evolutes can have a point of inflection.*

At a point of inflection on $r = F(\theta)i + G(\theta)j$, there must exist the condition that

$$\frac{d\theta}{dF} = \frac{d}{d\theta}\frac{(\arctan G'/F')}{F'} = \frac{(F'G'' - G'F'')}{(F')^3} = 0.$$

But since $\theta = \arctan G'/F'$, $(F')^2 + (G')^2 = F'G'' - G'F''$, so that for a point of inflection to exist we must have $1/F' + (G'/F')^2(1/F') = 0$. By Theorem I, however, F' cannot be infinite; therefore there can be no point of inflection on r.

Since F_1 and G_1 and all their derivatives with respect to θ are finite, the same method of reasoning shows that r_1 cannot have a point of inflection; likewise for all the other evolutes of r.

THEOREM III. *A plane curve with one infinite branch, no point singularities and continuous evolutes, cannot be symmetrical with respect to a point.*

Since there is only one branch to the curve and no point singularities, if there be a point of symmetry it must lie on the curve. Let the point of symmetry be $P(a, b)$ on the curve $y = f(x)$; let y^* be the value of y at the point $(a+h, b+k)$ in which h is positive and real and k of either sign, and $^*y$ the value of y at the point $(a-h, b-k)$. Then, since y is an analytic function of x,

$$y^* = b + h\,dy/dx + \tfrac{1}{2}h^2\,d^2y/dx^2 + (1/6)h^3\,d^3y/dx^3 + \cdots$$
$$^*y = b - h\,dy/dx + \tfrac{1}{2}h^2\,d^2y/dx^2 - (1/6)h^3\,d^3y/dx^3 + \cdots$$

[1] If one should inquire, for example, whether $4a^3x = y^4$ is of type C, he would find from $x = at^4/4$, $y = at$ that $dy/dx = 1/t^3$, whence $G = a\cot^{1/3}\theta$, $G' = -(1/3)\cot^{-2/3}\theta\csc^2\theta$. Therefore G' is infinite for $\theta = \pi/2$. Hence this is not a curve of type C.

For symmetry with respect to P, $b-{}^*y=y^*-b$, whence $d^2y/dx^2=0$. But this makes P a point of inflection; hence there can be no such point on the curve and a curve of type C cannot be symmetrical with respect to a point.

THEOREM IV. *A curve of type C crosses an axis of symmetry once and only once.*

For $r=Fi+Gj$ to be symmetrical with respect to the i-axis, F must be an even function of θ and G an odd function of θ. The curve must cross the axis at least once or it will have two infinite branches. Where the curve crosses the axis of symmetry it must cross at right angles, else there will be a singular point at the point of crossing. Suppose the curve crosses the axis of symmetry at A. If it again crosses at B, it must cross there at an angle of 90° with the axis and the arc AB will correspond to a symmetrical arc on the opposite side of the axis, and the curve will be entire and finite or else discontinuous. It therefore crosses the axis of symmetry once and only once.

THEOREM V. *A curve of type C and all its evolutes are symmetrical with respect to the same axis.*

Since F is an even function of θ and G an odd one, G', F'', G''', F'''' etc. are even functions while F', G'', F''', G'''' etc. are odd functions of θ. An inspection of the equations given in the proof of Theorem I shows that F_1, F_2, F_3, F_4, etc. are even functions of θ and G_1, G_2, G_3, G_4, etc. are odd ones, whence it follows directly that all the evolutes of a plane curve symmetrical with respect to an axis are symmetrical with respect to the same axis.

THEOREM VI. *A curve of type C cannot at any point be parallel to the axis of symmetry but may approach parallelism as it recedes towards infinity.*

As θ decreases from 90° at the point where the curve crosses the axis, if the curve should at any point become parallel to the axis of symmetry, it must at some succeeding point either cross the axis (a second time) or pass through a point of inflection, the result in either case being contrary to the previous theorems. Since θ may vary from 90° to 0° and since $d\theta/dF<0$, the curve may approach parallelism with the axis as θ approaches 0°. We shall suppose that θ varies from 90° to A and from $-90°$ to $-A$ where $0<A<90°$.

THEOREM VII. *The odd ordered evolutes of a curve of type C have cusps on the axis of symmetry and the even ordered evolutes cross the axis of symmetry at right angles at points of greatest absolute curvature.*

We assume the arcs to increase on all curves in either direction from the point on the axis of symmetry and the tangent directed in the sense of increase of the arc s. This means that, starting from the point on the axis of symmetry,

s is measured positively for r into both the first and fourth quadrants as θ varies from 90° to A and from $-90°$ to $-A$. Now we have

$$\tan\theta = G'/F' = -F_1'/G_1' = G_2'/F_2' = \cdots = G_{2k}'/F_{2k}' = -F_{2k+1}'/G_{2k+1}'$$

whence $\theta=\theta_1\pm 90°=\theta_2\pm 180°=\theta_3\pm 270°$, the upper signs going with the range 90° to A and the lower signs with the range $-90°$ to $-A$. In general

$$\theta_{4n} = \theta_{4n+1} \pm 90° = \theta_{4n+2} \pm 180° = \theta_{4n+3} \pm 270° \quad (n = 0,1,2, \cdots).$$

As θ varies from 90° to A, θ_{4n} varies from 90° to A; θ_{4n+1} varies from 0° to $A-90°$; θ_{4n+2} varies from $-90°$ to $A-180°$; and θ_{4n+3} varies from 180° to $A+90°$.

As θ varies from $-90°$ to $-A$, θ_{4n} varies from $-90°$ to $-A$; θ_{4n+1} varies from 0° to $90°-A$; θ_{4n+2} varies from 90° to $180°-A$; and θ_{4n+3} varies $-180°$ to $-90°+A$.

An inspection of these relations shows that, starting from the point on the axis of symmetry, even ordered evolutes extend perpendicularly to the axis in opposite directions, that is, cross it at right angles. Odd ordered evolutes, starting from the point on the axis of symmetry, extend in the same direction (parallel to the axis) and therefore have cusps at this point. It follows, therefore, that corresponding to the cusps on the odd ordered evolutes are points of greatest absolute curvature (vertices) on the even ordered evolutes.

THEOREM VIII. *Each evolute of a curve of type C of odd order has a point in common at its cusp with the evolute of next higher order at its vertex.*

Since $F_{2k}=F_{2k-1}-G_{2k-1}'$ and $G_{2k}=G_{2k-1}-F_{2k-1}'$ $(k=1, 2, \cdots)$, and since at a cusp on r_{2k-1}, F_{2k-1}' and G_{2k-1}' are each zero, it follows for this point on the axis of symmetry that $F_{2k}=F_{2k-1}$ and $G_{2k}=G_{2k-1}$; that is the curves r_{2k-1} and r_{2k} have a point in common on the axis of symmetry. Since the absolute value of the curvature is a maximum at this point on r_{2k}, it corresponds to a vertex on the curve.

THEOREM IX. *Evolutes of order $4n$ and $4n+3$ are concave in the same sense as the given curve of type C while evolutes of order $4n+1$ and $4n+2$ are concave oppositely to the given curve.*

An inspection of the variations in θ_{4n} $(n=0, 1, 2, \cdots)$ and in θ_{4n+3} as given in Theorem VII shows that for these evolutes the concavity is in the same direction as for θ, while the variations in θ_{4n+1} and in θ_{4n+2} are in the opposite direction. Taking the axis of symmetry as a half line directed positively from the vertex of the given curve as initial line for θ, we may say that the concavity of evolutes of order $4n$ and $4n+3$ is positive while that of evolutes of order $4n+1$ and $4n+2$ is negative.

ON *MÛLA*, THE HINDU TERM FOR "ROOT"

By BIBHUTIBHUSAN DATTA, University of Calcutta

The May (1926) issue of this Monthly contains an interesting article by Mr. Solomon Gandz on the origin of the term "root." It is interesting, no doubt, to follow the steps by which most of the civilized nations of the world came to use in the theory of numbers a term which essentially belongs to the vegetable kingdom. But certain statements in that article require correction. The learned writer has said: "The Chinese, indeed, do use the word *kun* to mean root, grass, and shrub, and the Hindus also use the word *mûla* for the root of a plant, but this was very likely due to the Arabic influence, which is so often seen in China and which may have spread into India by way of China." The latter part of this observation is certainly far from the truth.

Mr. Gandz together with Professor Smith[1] holds the opinion that the term "root" has its origin in the Arabic term *jadhr*. Originally *jadhr* was a concrete concept meaning "square basis." The earliest Arabic mathematician to give a clear definition of *jadhr* was Mohammed ibn Mûsâ al-Khowârizmî (*c.* 825). He "was possibly influenced by his Hebrew predecessors," who used the term *iqqar* probably in the same concrete sense. In later times, the original significance of *jadhr* was, however, forgotten by the Arabic scholars themselves. About 1600, they understood by it only an abstract concept. To express this misunderstood concept the medieval Latin translators of the Arabic works used the term *radix* ("root").[2] All these have been well pointed out by the learned writer. But he has fallen into an error in speaking of the Hindu mathematics.

So far as our present knowledge goes, the Hindus have used the term *mûla*[3] in connection with the theory of numbers, from more than a millenium before

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[1] David Eugene Smith, *History of Mathematics*, vol. 2, p. 150.

[2] Some early Latin translators of Al-Khowârizmî's works such as John of Sevile (1140), Robert of Chester (1145), and Gerard of Cremona (1150) used *radix* for *shai*, the Arabic term for the first power of the unknown quantity in an algebraic equation. As has been already remarked by Professor Karpinski, this use is unique. It seems that Al-Khowârizmî's definition is somewhat responsible for this usage. "Of these then the root is anything composed of units which can be multiplied by itself, or any number greater than unity multiplied by itself: or that which is found to be diminished below unity when multiplied by itself. The square is that which results from the multiplication of a root by itself." (See Robert of Chester's Latin translation of the *Algebra of Al-Khowârizmî*, edited with English translation by Louis Charles Karpinski, New York, 1915, p. 66). It is not easy to say why Al-Khowârizmî defined the term for the unknown quantity by the property that it can be multiplied by itself. But such a definition is bound to lead others to regard the unknown quantity as the root. At any rate it discloses an abstract concept of the Arabic term for root.

[3] The term *mûla* is also used with a different significance in the Hindu mathematics. It means the capital of a loan as opposed to the interest. But our observations in this article will always remain confined to its use in connection with the square root and the cube root. It may be pointed out that the former use of the word *mûla* is also an old one.

the Arabic *jadhr* became transformed into the Latin *radix*. It is found in the works of the earliest known Hindu mathematician, Âryabhaṭa who wrote in 499 A.D., that is, three centuries and a quarter before Mohammed ibn Mûsâ. It was also used by Varâhamihira (505), Lalla (c. 578), Brahmagupta (628), Śrîdhara (c. 750), Mahâvîra (c. 853), Âryabhaṭa the younger (c. 950), and Bhâskara(1150). It is noteworthy, indeed, that none of the Hindu scholars has ever attempted to define the term *mûla*. They began with clear definitions of the terms *varga* (square) and *ghana* (cube). Here they used more or less identical words: "the product of two equal (quantities) is a square," "the product of three equal (quantities) is a cube."[1] Immediately after the definitions, the Hindu mathematicians proceeded to expound the rules for extracting the *mûla* of a square and of a cube. So for them the term *mûla* must have been derived directly from the reverse process. This will be further corroborated by the following statement of Brahmagupta which was made on a different occasion: "The root of a square is that of which it is the square."[2] Thus we find that the Hindus used the term *mûla*, in the theory of numbers, as an abstract concept from the fifth century A.D. Hence Mr. Gandz's conjecture that the Hindus did so under the Arabic influence must be wrong.

Synonymous with the term *mûla*, the Hindus use another term in the theory of numbers, namely *pada*.[3] The word *pada* has been used with this significance by almost all the Hindu mathematicians with the exception of Âryabhaṭa the elder. Hence the use of *mûla* is older than the use of *pada* in this respect. Now *pada* is not certainly a botanical term. In fact, it has very little to do with the vegetable kingdom.

In Sanskrit—as also in Pali—the word *pada* means 'step,' 'footing,' 'part, portion,' 'side,' 'place,' 'cause,' 'a square on a chess-board,' 'a plot of ground,' etc. The word *mûla* has various and wider meanings. Among others it means 'root' (of a plant or tree; but also figuratively the foot or lowest part or bottom of anything), 'basis,' 'foundation,' 'cause,' 'origin,' 'the edge' (of the horizon),

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[1] Brahmagupta has also terms for higher powers of a number. His term for the continued product of four or more like quantities is *tad-gata* literally, 'raised to that'; thus *pañca-gata* means fifth power—literally, 'raised to the fifth (power)'; *ṣaḍ-gata* means sixth power. The higher powers of a number are also expressed by a combination of *varga* and *ghana*. Evidently the latter mode can have only limited applications. See *Brâhma-sphuṭa-siddhânta*, ed. Sudhakara Dvivedi, Benares (1902), Ch. xviii, verse 42.

[2] *Padam kṛtir-yat tat. Brâhma-sphuṭa-siddhânta*, Ch. xviii, verse 35.

[3] In the Hindu mathematical treatises, the word *pada* has also been used to denote the number of terms in a series in arithmetical progression. It is very likely that this application arose out of the analogy with the particular result, $1+3+5+\cdots$ to n terms $=n^2$; so that $n=\sqrt{\text{sum}}$. Cf. Śrîdhara's *Triśatikâ*, ed. Sudhakara Dvivedi, Benares, verse 11, where the term *varga* has also been defined as the sum of this particular series in arithmetical progression.

and 'immediate neighborhood.'[1] Certain meanings of either of the two words must be figurative. The import common to the two are 'cause,' 'origin,' 'foundation,' or 'basis.' These are therefore the concepts embodied in the terms *mûla* and *pada* in the theory of numbers. Again it is very difficult to say which meaning of the word *mûla*, botanical or otherwise as 'cause,' etc., is the older. The word was used with each significance in the earliest Vedic literature of the Hindus; and, indeed, by the time it was introduced into the Hindu mathematics, it had acquired various other meanings. But it is also a fact that the use of the word *mûla* in the abstract sense of 'cause,' 'origin,' 'foundation,' 'basis' has still wider and more copious applications in the Sanskrit and Pali literatures as well as in the Hindu theological and philosophical terminologies. So it is natural to assume as we have already done from other considerations, that the Hindus introduced the word *mûla*—like the word *pada*—into the theory of numbers in the abstract sense of 'cause,' 'origin,' 'foundation,' or 'basis.'

There is, however, one Hindu mathematician, namely Âryabhaṭa the elder, who seems to have in view also a concrete concept for the term *mûla*, for he gives geometrical definitions for the terms *varga* and *ghana* side by side with the arithmetical ones. He says: "*Varga* is the equi-four-sided figure (*samacaturaśra*), the area (*phala*) and the product of two equal (quantities); similarly *ghana* is the (equi-) twelve-sided figure (*dvâdaśâśra*), the volume (*phala*) and the product of three equal (quantities)."[2] Hence the *mûla* of the *varga* must mean the cause or basis of the square figure, that is, the side; the cause or basis of the square area, that is, square unit; and the cause or basis of the square number, that is, a pure number. Similarly the *mûla* of the *ghana* must mean the side, the unit cube, as also a pure number. From what has been said before, it will appear that these interpretations as the side or the edge, the square unit, or the unit cube, are not foreign to the words *mûla* and *pada*. It is to be noticed, however, that the geometrical significance contained in the above definitions of Âryabhaṭa appears to have been overlooked by the later Hindu mathematicians. For we find nothing of the kind in the works of Brahmagupta, Śrîdhara, Mahâvîra and Bhâskara. Certain recognition was given to it in Prithudakaswâmi's commentary on *Brâhma-sphuṭa-siddhânta* (975).[3]

A clearer and fuller exposition is that of Parameśvara (c. 1430), a scholiast of the elder Âryabhaṭa. In commenting upon the above definition of *varga*, he

[1] See Monier-Williams's *Sanskrit-English Dictionary*, new edition by Leumann and Cappeller; Rhys Davids and Steele's *Pali-English Dictionary*. Cf. also the St. Petersburg Sanskrit dictionary.

[2] *Âryabhaṭîya, Gaṇitapâda*, verse 3. The term *samacaturasra* is a very old one; literally it means equi-four-angled figure, for *asra* = angle. But it is used to denote a square figure. Similarly *dvâdaśâsra* is used to denote a cube.

[3] See Henry T. Colebrooke, *Algebra with Arithmetic and Mensuration from The Sanscrit of Brahmegupta and Bhascara*, London (1817), pp. 280–1, footnotes.

remarks: "The summation of the *kṣetraphalas* (meaning the elementary areas) is termed the *varga*. In the case of a given square, the *kṣetraphalas* are the squares formed by four sides measuring one cubit each, that are contained inside it. Similarly, in the case of a triangle, a circle, or any (non-square) figure also, the *kṣetraphalas* are the square pieces formed by four sides measuring one cubit each, that can be assumed to be contained inside it." Another explicit statement is that of Kamalākara, the author of the *Siddhānta-tattva-viveka*. According to him "the square root (*mûla*) is of two kinds, a line and a number." His further statements deserve more than a passing notice. He says that the root of a perfect square number is a line as well as a number but the root of an imperfect square number, that is, the value of a surd is only a line. He has explained his ideas with the help of illustrative examples and by evaluating certain surds geometrically.[1]

It may be remarked in conclusion that what is found in the geometry section of the works of Mohammed ibn Mûsâ is an explicit statement and fuller treatment of one of the above mentioned geometrical ideas of Âryabhaṭa, viz., that of the unit square. Though Âryabhaṭa's ideas appear less explicit,[2] they are certainly wider and more logical. For his ideas about the *mûla* include the Greek conception of 'side,' the so-called Arabic conception of 'area' as well as arithmetical conception of pure number. Further it includes the conception of a unit cube which is found nowhere else.[3] As Al-Khowârizmî was acquainted with the Hindu mathematics, particularly with the *Gaṇita* of Âryabhaṭa, is it possible that he was influenced by his Hindu predecessor.[4]

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[1] *Siddhânta-tattva-viveka*, ed. Sudhakara Dvivedi, Benares (1885), Ch. iii, verse 21 (*vâsanâbhâṣya*); cf. also Ch. ii, verse 158.

[2] The extreme conciseness of Âryabhaṭa's book is probably responsible for it.

[3] It may be noted in passing that Bhâskara's measures include definitions of unit cube (*ghana-hasta*, literally solid cubit) and also unit area (*nivartana*).

[4] Mohammed ibn Mûsâ al-Khowârizmî wrote a book, *On the Hindu Art of Reckoning*. His algebra contains a reference to Âryabhaṭa's value of π.

The definition of *iqqar*, the Hebrew word for *jadhr*, quoted by Mr. Gandz from an old Hebrew geometry by an anonymous writer, is vague and almost meaningless—as has also been admitted by him. He had to amend it in the light of the concrete meaning of Al-Khowârizmî, so as to bring it into line with the definition of the latter writer. Moreover, the date of this anonymous Hebrew writer is so uncertain that it is difficult to determine the chronological sequence between him and Mohammed ibn Mûsâ (Smith, *History*, vol. 1, p. 174). So it cannot be stated definitely who was influenced by whom. Dr. Gandz thinks "perhaps the two were influenced by some common source which is now unknown." It is not impossible that the works of Âryabhaṭa were this common source. For we know it on the authority of the Bishop Sêverus Sebokht of Nisibus (in Northern Mesopotamia) that the knowledge of the Hindu science of arithmetic had spread into Syria as early as the beginning of the seventh century A. D.

QUESTIONS AND DISCUSSIONS

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The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the separate department of Problems and Solutions.

DISCUSSIONS

I. CONCERNING A THEOREM IN DETERMINANTS

By J. J. NASSAU, Case School of Applied Science

The following is a proof of a theorem stated in this Monthly by Professor H. S. Uhler:[1]

The result of compounding the matrix

$$M_b = \begin{pmatrix} b_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ b_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ b_k & a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}; \; k \leqq (n-1), \; rank = k$$

with its conjugate is a square matrix. The determinant D_b of this matrix is[2]

$$\begin{vmatrix} b_1^2 + c_{11} & b_1b_2 + c_{12} & \cdots & b_1b_k + c_{1k} \\ b_1b_2 + c_{21} & b_2^2 + c_{22} & \cdots & b_2b_k + c_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ b_1b_k + c_{k1} & b_2b_k + c_{k2} & \cdots & b_k^2 + c_{kk} \end{vmatrix} = \begin{vmatrix} b_1 & a_{11} & a_{12} & \cdots & a_{1,k-1} \\ b_2 & a_{21} & a_{22} & \cdots & a_{2,k-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ b_k & a_{k1} & a_{k2} & \cdots & a_{k,k-1} \end{vmatrix}^2$$

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[1] Let $-N$ denote the bordered symmetrical determinant

$$\begin{vmatrix} 0 & b_1 & b_2 & \cdots & b_k \\ b_1 & c_{11} & c_{12} & \cdots & c_{1k} \\ b_2 & c_{21} & c_{22} & \cdots & c_{2k} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ b_k & c_{k1} & c_{k2} & \cdots & c_{kk} \end{vmatrix}$$ where $c_{rs} \equiv \sum_{j=1}^{j=n} (a_{r,j}\, a_{s,j}) = c_{sr}$ and the a's belong

to the matrix $M = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{1n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}$; $k \leqq (n-1)$, rank $= k$.

Is there a simple proof for the following theorem? *N equals the arithmetical sum of the squares of $\binom{n}{k-1}$ determinants of order k in every one of which the first column read downward consists of $b_1, b_2, \cdots, b_k$ and the remaining $k-1$ monomial columns constitute collectively one of the combinations of the n columns of the matrix M taken $k-1$ at a time.*

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[2] Scott and Mathews *Theory of Determinants*, p. 54.

$$+\begin{vmatrix} b_1 & a_{11} & \cdots & a_{1,k-2} & a_{1k} \\ b_2 & a_{21} & \cdots & a_{2,k-2} & a_{2k} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ b_k & a_{k1} & \cdots & a_{k,k-2} & a_{kk} \end{vmatrix}^2 + \cdots + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix}^2$$

$$+\begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1,k+1} \\ a_{21} & a_{23} & \cdots & a_{2,k+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{k1} & a_{k3} & \cdots & a_{k,k+1} \end{vmatrix}^2 + \cdots .$$

The left-hand side can be written as the sum of 2^k determinants of which $\binom{k}{2}+\binom{k}{3}+\cdots+\binom{k}{k}$[1] determinants have two or more columns alike. Hence the left-hand side composed of the $k+1$ remaining determinants is

$$\begin{vmatrix} b_1^2 & c_{12} & \cdots & c_{1k} \\ b_1b_2 & c_{22} & \cdots & c_{2k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ b_1b_k & c_{k2} & \cdots & c_{kk} \end{vmatrix} + \begin{vmatrix} c_{11} & b_1b_2 & c_{13} & \cdots & c_{1k} \\ c_{21} & b_2^2 & c_{23} & \cdots & c_{2k} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ c_{k1} & b_2b_k & c_{k3} & \cdots & c_{kk} \end{vmatrix} + \cdots$$

$$+\begin{vmatrix} c_{11} & \cdots & c_{1,k-1} & b_1b_k \\ c_{21} & \cdots & c_{2,k-1} & b_2b_k \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ c_{k1} & \cdots & c_{k,k-1} & b_k^2 \end{vmatrix} + |c_{11}\ c_{22}\ \cdots\ c_{kk}| =$$

$$-\begin{vmatrix} 0 & b_1 & b_2 & \cdots & b_k \\ b_1 & c_{11} & c_{12} & \cdots & c_{1k} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ b_k & c_{k1} & c_{k2} & \cdots & c_{kk} \end{vmatrix} + |c_{11}\ c_{22}\ \cdots\ c_{kk}|.$$

The right-hand side is made up of the sum of $\binom{n}{k-1}$ determinants of the type $|b_1a_{21}\cdots a_{kk-1}|^2$ and $\binom{n}{k}$ determinants of the type $|a_{11}a_{22}\cdots a_{kk}|^2$.

This last sum of $\binom{n}{k}$ determinants is equivalent to the determinant of the square matrix obtained by compounding the matrix

$$\begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{Vmatrix}$$

with its conjugate, and hence is equal to $|c_{11}c_{22}\cdots c_{kk}|$.

[1] The symbol $\binom{\kappa}{l}$ means the number of combinations of k things l at a time. It is well known that $\binom{k}{1}+\binom{k}{2}+\binom{k}{3}+\cdots+\binom{k}{k}=2^k$.

From the above, the proof of the theorem is evident.

REMARKS: 1. The above proof suggests the relation

$$D_b = -\begin{vmatrix} -1 & b_1 & \cdots & b_k \\ b_1 & c_{11} & \cdots & c_{1k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ b_k & c_{k1} & \cdots & c_{kk} \end{vmatrix}$$

2. If in the determinant $\begin{vmatrix} 0 & b_1 & \cdots & b_k \\ b_1 & c_{11} & \cdots & c_{1k} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ b_k & c_{k1} & \cdots & c_{kk} \end{vmatrix}$ we let $b_1 = b_2 = \cdots = 1$,

we have the following theorem:

THEOREM: *The sum of the signed primary minors*[1] *of the symmetrical determinant* $|c_{11}c_{22} \cdots c_{kk}|$ *is equal to the sum of the squares of* $\binom{n}{k-1}$ *determinants of order k in each of which the first column is composed of ones and the remaining* $k-1$ *monomial columns are composed collectively of one of the combinations of the n columns of the matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}$$

taken $k-1$ *at a time.*

II. A SIMPLE WAY TO DISCUSS POINTS OF INFLECTION ON PLANE CUBIC CURVES

ALAN D. CAMPBELL, Syracuse University

In this paper we show by means of illustrations a simple but very effective way to discuss points of inflection on plane cubic curves. We used this method with excellent results in a paper[2] entitled *Plane cubic curves in the Galois fields of order* 2^n and in another paper (not yet published) on plane cubic curves in the Galois fields of order p^n, $p>2$. [We abbreviate the name of these fields to $GF(p^n)$, $p>2$]. The method can be used also in the real and the complex domains and it avoids the (often difficult) task of solving simultaneously a cubic and its Hessian. In the $GF(2^n)$ the Hessian of a cubic is the cubic itself multiplied by the coefficient of the xyz-term in the cubic (loc. cit p. 396); so here the type of discussion given below seems to be the only feasible one.

[1] Muir, Proceedings of the Royal Society of Edinburgh, vol. 24, Part IV, p. 390.

[2] Annals of Mathematics, vol. 27 (1926), pp. 395–406.

Our plan consists mainly in seeking the tangents at points of inflection and not the points of inflection themselves. (We call such points more briefly "inflections"). Let us consider the cubic

$$y^2z = x^2(x + z), \tag{1}$$

which has an inflection at (0, 1, 0) with $z=0$ as tangent and a crunode at (0, 0, 1) with $y=\pm x$ as tangents. The tangent at any other inflection must have an equation of the form $l \equiv z - ax - by = 0$. Solving this simultaneously with (1) we get the equation

$$x^3(1 + a) + bx^2y - axy^2 - by^3 = 0. \tag{2}$$

If l is to be a tangent at an inflection and (1) lies in the complex or the real domain or the $GF(p^n)$ with $p>3$, then (2) must be of the form

$$y^3 + 3cxy^2 + 3c^2x^2y + c^3x^3 = 0. \tag{3}$$

Note that these $GF(p^n)$ are the finite algebras corresponding to the finite geometries that have just p^n+1 points on each line.[1] In these algebras every number is reduced modulo p, and if $n>1$ a second modulus called a Galois imaginary is used. Thus in $GF(2^2)$ the five points on the line $x=0$ have the coordinates (0, 0, 1), (0, 1, 0), (0, 1, 1), (0, i, 1), (0, i+1, 1) where i is a root of the equation $x^2+x+1=0$ which has neither 0 nor 1 as a root. Also in $GF(2^n)$ we have $(\alpha x+\beta y+\gamma z)^2=\alpha^2x^2+\beta^2y^2+\gamma^2z^2$ and every number is a perfect square with just one square root. In $GF(p^n)$, $p>2$ there are non-squares. If $p=3$ we have $(\alpha x+\beta y+\gamma z)^3=\alpha^3x^3+\beta^3y^3+\gamma^3z^3$. In $GF(p^n)$, where $p^n=3m+1$, there are non-cubes, but in the other Galois fields every number has at least one cube root (and in some fields three cube roots). There are irreducible cubic equations in every $GF(p^n)$.

From (3) we see that we must have $(1+a)/(-b)=c^3$, $-1=3c^2$, $a/b=3c$; hence $c=\pm\sqrt{(-1/3)}$, $a=-3/(3+c^2)$, $b=-1/(3c+c^3)$. [The value $b=0$ gives $a=0$ and the tangent $z=0$ at (0, 1, 0)]. Therefore in the real domain the two other inflections on (1) are imaginary, but in the $GF(p^n)$, $p>3$, where $-1/3$ is a square, these inflections are real. (This method gives also the tangents at possible nodes or cusps).

In the $GF(2^n)$ the equation (3) must be replaced by

$$y^3 + cxy^2 + c^2x^2y + c^3x^3 = 0. \tag{3'}$$

From (3') we get $(1+a)/b=c^3$, $1=c^2$, $a/b=c$, since $-1\equiv+1$ modulo 2. Hence $c=1$, $a=b$, $1+a=b$ which are incompatible equations; so if $p=2$ the cubic (1) has no more inflections.

In the $GF(3^n)$ we must replace (3) by $y^3+c^3x^3=0$; so we must have $(1+a)/(-b)=c^3$, $-1=0$, $a/b=0$ (which are impossible equations); hence for $p=3$ the cubic (1) has just the one inflection (0, 1, 0). The above facts can be obtained easily (except for $p=2$) by solving the Hessian of (1) simultaneously

[1] See Veblen and Young *Projective Geometry* vol. 1, pp. 201, 202 and also L. E. Dickson's *Linear Groups*, pp. 1–40.

with (1); but the conditions for the existence or nonexistence of other real inflections on (1) appear more readily by the above process.

Next let us consider the cubic

$$y^2z = x^3 + (\alpha + 1)x^2z + \alpha xz^2, \qquad \alpha \neq 0 \text{ or } 1, \tag{4}$$

which has an inflection at (0, 1, 0) with $z=0$ as tangent. If we solve $y=ax+bz$ simultaneously with (4) we get

$$x^3 + (-a^2 + \alpha + 1)x^2z + (\alpha - 2ab)xz^2 - b^2z^3 = 0.$$

In the real or the complex domain and in the $GF(p^n)$, $p>3$ this equation must be of the form (3) with x and z replacing y and x respectively. Hence we have $-a^2+\alpha+1=3c$, $\alpha-2ab=3c^2$, $-b^2=c^3$, giving us the quartic

$$3c^4 - 4(\alpha + 1)c^3 + 6\alpha c^2 - \alpha^2 = 0 \tag{5}$$

with resolvent cubic[1]

$$\theta^3 + (1/108)(\alpha^4 - 2\alpha^3 + \alpha^2) = 0. \tag{6}$$

Dickson shows[2] that, if (6) has one and only one root in a $GF(p^n)$, $p>3$, then (5) is irreducible and so (4) has no more real inflections; and if (6) is irreducible then (5) has just one real root and so (4) has two more real inflections (if $-a^2+\alpha+1=3c$ and $b^2=c^3$ have real roots in a and b).

In the $GF(3^n)$ we have for the cubic (4) $-a^2+\alpha+1=0$, $\alpha-2ab=0$, $-b^2=c^3$; hence $a=\pm(-\alpha-1)^{1/2}$, $b=\alpha/\pm 2(-\alpha-1)^{1/2}$; so that according as $-\alpha-1$ is (or is not) a perfect square the cubic has two more real (or imaginary) inflections. In the $GF(2^n)$ we have for the cubic (4) $-a^2+\alpha+1=c$, $\alpha=c^2$, $-b^2=c^3$; hence $a=\alpha^{1/2}+\alpha^{1/4}+1$, $b=\alpha^{3/4}$ give us the tangent at one more real inflection.

Finally let us consider the cubic

$$y^3 + \alpha z^3 + xyz = 0, \qquad \alpha \neq 0, \tag{7}$$

which has a crunode at (1, 0, 0) with $y=0$ as one of the tangents. If we solve (7) simultaneously with $x=ay+bz$, we get $y^3+ay^2z+byz^2+\alpha z^3=0$. If α is not a cube, we have no real inflections on (7). If $p=3$ the cubic has just one real inflection with tangent $x=0$. In the real domain we have one real and two imaginary inflections on (7).

In the above mentioned finite geometries we find plane cubic curves with numerous peculiarities, such as cubics with no node or cusp or real inflection,[3]

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[1] See Burnside and Panton *Theory of Equations*, 2nd. edition, p. 127.

[2] Bulletin of American Mathematical Society, vol. 13 (1906), p. 5.

[3] Annals of Mathematics, loc. cit., pp. 398–400.

cubics with nine real inflections (such as $z^3+x^2y+xy^2=0$ when $p=2$, $n>1$), cubics with just one real inflection (see above), and so forth. These peculiarities are well brought out by the method (discussed in this paper) of finding the tangents at inflections.

III. A Note on the Function $y=x^x$

By Raymond Garver, University of Rochester

I consider the function $y=a^x$, $a<0$, mentioned by Professor Campbell in the April Monthly, particularly useful because it illustrates that not all functions can be represented graphically. This is a novel idea to most students who have had analytic geometry and a beginning course in calculus, and one which I believe is worth-while bringing out. It might also be well to list other conditions (besides discontinuity at an infinite number of points) which render a function incapable of graphical representation.

A somewhat similar and equally interesting function is $y=x^x$. For $x>0$, the function is continuous and differentiable, and if we define the function as 1 for $x=0$, it is also continuous on the right for this value. The right-hand derivative does not exist, however, for $x=0$; at least it does not remain finite. The function has a minimum value for $x=1/e$, and the corresponding functional value is approximately .69. The area bounded by the curve, two positive ordinates, and the x-axis is given by $\int_a^b x^x dx$. I have not been able to evaluate the indefinite integral, but the definite integral can of course be approximated. On the other hand, for $x<0$, the function is completely discontinuous, much as the function a^x, $a<0$.

More generally, we might consider functions of the type $y=f(x)^{\varphi(x)}$. Almost every special case will be found to give interesting results.

IV. On the Approximate Division of the Circle[1]

By Roger A. Johnson, Hunter College

A recent note in this Monthly[2] discusses a construction for the approximate division of the circle into any number of equal parts, said to be used by draftsmen. The purpose of the present note is to call attention to a very similar construction, no more complicated but a much closer approximation.

The striking aspect of these constructions is the fact that one and the same method is applicable to all cases and yields a fairly accurate result. In the second

[1] Editor's Note: The construction given in this paper is credited to the German Duke Karl Bernhard of Sachsen-Weimar. See K. T. Vahlen, *Konstruktionen und Approximationen*, (Teubner, 1911), p. 305.

[2] E. J. McShane, *An approximate division of the circle*, this Monthly, vol. 34 (1927), p. 140.

construction introduced below, the theoretical error is less than the inaccuracy of ordinary drawing.

The construction discussed by Messrs. Gwinner and McShane is as follows:

Let O be the center, AB a diameter, of a circle whose circumference is to be divided into n equal parts. Let ABP be an equilateral triangle; divide AB into n equal parts, and let C be the second point of division from A. Let PC produced meet the circle at D; then arc AD is nearly the nth part of the circumference.

The following construction is given in the *Exercise Manual in Geometry of Wentworth and Hill* (Ginn & Co., 1883) without references or comments.

With the same data as before, let AB be divided into n equal parts; and extend OA to E, so that AE equals one of these parts. Draw OF perpendicular to AB and equal to OE; let the first intersection of EF with the circle (i.e., that which is nearer to E) be G. Let H be the third point of division from A; then GH is approximately equal to a side of a regular polygon of n sides.

It will be seen that neither of these approximations is very close when n is 5, and that both are exact when $n=6$. For values of n greater than 6, the second method is uniformly a much closer approximation than the first. Mr. McShane finds that in the first method the error remains less than the value 10.265%, which it approaches as n increases; it would seem that this persistent high percentage of error renders the method less interesting from a theoretic standpoint. We can show that in the second method the error tends toward a limiting value of approximately 2/3 of 1% as the number of sides increases.

Without much difficulty we find that in the first solution the result, taking the radius OA as unity, is

$$\sin\theta = \frac{3^{1/2}[(1+16n^{-1}-32n^{-2})^{1/2}-(1-4n^{-1})^2]}{4-8n^{-1}+16n^{-2}}$$

and in the second

$$GH = [1-8n^{-1}+48n^{-2}-(1-6n^{-1})(1-4n^{-1}-4n^{-2})^{1/2}].$$

It is not easy to compare these results with the true values $\sin(2\pi/n)$ and $2\sin(\pi/n)$ respectively, except by actual computation. The following numerical values may be of interest:

n	8	10	16	24	50
True central angle	45°	36°	22°30′	15°	7°12′
Angle by first method	45°11′	36°21′	23°6′	15°38′	7°40′
Angle by second method	44°57.2′	35°56′	22°28.4′	15°0.3′	7°13.3′

If we expand the formulas according to increasing powers of $1/n$, we have in the respective cases

$$\sin\theta = 3^{1/2}\left(\frac{4}{n}-\frac{8}{n^2}\cdots\right); \qquad \overline{GH}^2 = 40^{1/2}\left(\frac{1}{n}-\frac{1}{5n^2}\cdots\right).$$

For sufficiently large values of n we may compare the accuracy of the solutions by comparison of the first terms of these expansions with $\sin(2\pi/n)$ or merely with $(2\pi/n)$. In the first method, as obtained by McShane, the ratio of the solution to the true value approaches as limit the ratio

$$4\sqrt{3}/2\pi = 2\sqrt{3}/\pi = 1.10265$$

and in the second case,

$$\sqrt{40}/2\pi = \sqrt{10}/\pi = 1.0067 \ .$$

Thus there is a residual error in the first method of over 10% and in the second method of less than 1%. It would be interesting to know whether a method can be devised in which the error approaches zero with increasing values of n.

RECENT PUBLICATIONS

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

NEW BOOKS RECEIVED

EDITOR'S NOTE: Hereafter there will be published each month in this department a list of all new books received. The first of these lists, which appears below, includes books received for some months past for which reviews have not yet been published. Not all books received are suitable for review in this periodical; and it is not always possible to find some one competent and willing to write a review. For these reasons we can not guarantee to publish reviews of all books sent to us for that purpose, but titles of all books received will be printed monthly.

BARNETT, I. A. *Plane Analytic Geometry*. New York, John Wiley and Sons, 1926. ix+269 pages. $2.00.

BECK, H. *Einführung in die Axiomatik der Algebra*. Berlin, Walter de Gruyter, 1926. x+197 pages.

BERNARD, D. M. *Plane Geometry*. Richmond, Johnson Publishing Company, 1927. xv+334 pages. $1.24.

BERWICK, W. E. H. *Integral Bases*. (Cambridge tracts, No. 22) Cambridge, University Press, 1927. 95 pages. 6½ shillings.

BRIDGMAN, P. W. *The Logic of Modern Physics*. New York, The Macmillan Company, 1927. xiv+228 pages. $2.50.

BUCHANAN, H. E. and SPERRY, P. *Plane Trigonometry and Tables*. Richmond, Johnson Publishing Company, 1926. xi+116+112 pages. $1.64.

CARMICHAEL, R. D., DAVIS, H. T., MACMILLAN, W. D. and HUFFORD, M. E. *A Debate on the Theory of Relativity*. Chicago, Open Court Publishing Company, 1927. viii+154 pages. $2.00.

CLARK, J. K. and OTIS, A. S. *Modern Plane Geometry*. Yonkers-on-Hudson, World Book Company, 1927. x+310 pages. $1.36.

DALTON, O. M. *The Byzantine Astrolabe at Brescia*. London, Oxford University Press, 1927. 14 pages, 3 plates. $0.70.

DAVIDSON, J., and PRESSLAND, A. J. *A Second Geometry*. Oxford, The Clarendon Press, 1926. 128 pages. $0.85.

EINSTEIN, A. *Investigations on the Brownian Movement*. New York, E. P. Dutton and Company, 1926. viii+124 pages. $1.75.

FORDER, H. G. *The Foundations of Euclidean Geometry*. Cambridge, University Press, 1927. xii+349 pages. 25 shillings.

FORSYTH, A. R. *Calculus of Variations*. Cambridge, University Press, 1927. xxii+656 pages. 50 shillings.

FRAENKEL, A. *Zehn Vorlesungen über die Grundlegung der Mengenlehre*. Leipzig, B. G. Teubner, 1927. x+182 pages.

GRASSMAN, H. *Projektive Geometrie der Ebene*. Zweiter Band, Zweiter Teil. Leipzig, B. G. Teubner, 1927. xvi+522 pages. 20 marks.

HAUSDORF, F. *Mengenlehre*. Zweiter Auflage. Berlin, Walter de Gruyter, 1927. 285 pages.

HUDSON, HILDA P. *Cremona Transformations in Plane and Space*. Cambridge, University Press, 1927. xx+454 pages.

IGNATOWSKY, W. *Die Vektoranalysis und ihre Anwendung in der Theoretischen Physik*. Dritte Auflage Teilen I and II. Leipzig, Teubner, 1926.

INCE, E. L. *Ordinary Differential Equations*. London, Longmans, Green, and Company, 1927. viii+558 pages. $12.00.

KAFKA, H. *Die Ebene Vektorrechnung und ihre Anwendung in der Wechselstromtechnik. Teil 1, Grundlagen*. Leipzig. Teubner, 1926. viii+132 pages.

KARSTEN, K. G. *Charts and Graphs*. New York, Prentice-Hall, Inc., 1923. xl+734 pages. $6.00.

KEASEY, M. A., KLINE, G. A., and MC ILHATTEN, D. A. *Plane Trigonometry, with Tables*. Philadelphia, P. Blakiston's Son and Company, 1927. vii+130 pages. $1.28.

KUHN, H. W. and MORRIS, C. C. *The Mathematics of Finance*. Boston, Houghton Mifflin Company, 1926. xiii+340 pages. $3.00.

LOVE, A. E. H. *The Mathematical Theory of Elasticity*. Fourth Edition. Cambridge, University Press, 1927. xviii+643 pages.

MULLINS, G. W. and SMITH, D. E. *Freshman Mathematics*. Boston, Ginn and Company, 1927. vi+386 pages. $3.00.

PARSONS, G. L. *Elementary Integral Calculus*. Cambridge, University Press, 1926. ix+127 pages.

REYMOND, ARNOLD. *History of the Sciences in Greco-Roman Antiquity.* Translated by Ruth Gheury de Bray. New York, E. P. Dutton and Company, 1927. x+245 pages. $2.50.

RIETZ, H. L. *Mathematical Statistics* (The third Carus Mathematical Monograph). Chicago, Open Court Publishing Company, 1927. xi+181 pages.

SOMMERFELD, A. *Three Lectures on Atomic Physics.* (Trans. by A. L. Brose.) New York, E. P. Dutton and Company, 1926. 70 pages.

STEFFENSEN, J. F. *Interpolation.* Baltimore, The Williams and Wilkins Company, 1927. ix+248 pages. $8.00.

TREVOR, J. E. *The General Theory of Thermodynamics.* Boston, Ginn and Company, 1927. x+104 pages. $1.60.

WALMSLEY, C. *Mathematical Analysis.* Cambridge, University Press, 1926. x+293 pages.

WASSERLOOS, EWALD and WOLFF, GEORG; Editors. *Mathematisch-Naturwissenschaftlich-Technische Bucherei.* Band 1, 2, 3, 5, 6. Berlin, Otto Salle, 1927. 142, 120, 75, 62, 122 pages.

WEATHERBURN, C. E. *Differential Geometry of Three Dimensions.* Cambridge, University Press, 1927. xii+268 pages.

WHITEHEAD, A. N. and RUSSELL, BERTRAND. *Principia Mathematica,* Vols. 1, 2, 3. Second edition. Cambridge, University Press, 1925–1927. 45 shillings, 45 shillings, and 25 shillings.

REVIEWS

Statistical Method in Educational Measurement. By ARTHUR S. OTIS. Yonkers, N.Y., World Book Company, 1925. xi+337 pages.

Serious difficulties face the scientific workers in the field of Education who have introduced *measurement* as the basis for improving methods of teaching. In the first place, while the statistical theory essential for correct interpretation of measurements is as yet somewhat imperfect, it is nevertheless necessary to apply the theory in its present state to meet immediate needs. In the second place, useful application of the statistical theory of measurement must be made by large numbers of teachers and administrators of slight mathematical and statistical training. They must therefore be provided with simple working rules, charts, and "common-sense" explanations of principles. In the third place, the considerable volume of experience which has been built up by the use of the simple approximate methods constitutes somewhat of an obstacle to the refinements which ought now to be inaugurated, both in order to make possible further progress in the fields in which the new point of view has been accepted, and to facilitate the entrance into new fields (e.g., college grading) in which the scientific inadequacy of the approximate methods is one cause of the opposition encountered.

In this new text-book, Dr. Otis attacks very successfully the difficulties of the second type. As he says in his preface:

"The purpose of the book is to present the subject in such a way that it can be understood by those who know nothing whatever about statistical method. · · · Several new charts for practical use are introduced. · · · Indeed, throughout the book the practical application of methods has been kept in mind, and only such discussion of theory is given as is thought necessary to make the use of the methods and devices intelligent rather than rule-of-thumb."

Nevertheless, while stressing the simple working procedure, Dr. Otis avoids the third difficulty mentioned above, by admitting frankly the approximate nature of many of the methods recommended, and even advocating improved methods in certain cases, as for instance in the computation of the "I.Q." The research worker who is primarily concerned with improvement in methods will find in many cases that a phrase or a paragraph of Dr. Otis opens the way for what he has in mind. For this reason the book may be expected to prove helpful to students of mathematical statistics who wish a practical background in the field of education for the mathematical discussion of frequency curves and correlation.

From the mathematical point of view, however, there are several points in which the treatment might well be modified somewhat. In his introductory discussion of "percentile curves," for instance, he would lead the mathematician more easily along familiar paths if these percentile graphs were related more directly to the areas of frequency curves, and the titles were stated more specifically—for example, "Per cent of cases in which the test score exceeded that indicated on base line." It should be noted that if the diagram implied by the idea of "area" or by the title just stated is turned through 90°, so that the base line becomes the right hand boundary, the result is exactly the form of the diagram recommended by Dr. Otis.

The use of the phrase "reliability coefficient" by Dr. Otis and other educational statisticians seems objectionable. He says (p. 256):

"The reliability of a test is the degree to which it is consistent in measuring that which it measures, but the validity of a test is the degree to which it measures that which it purports to measure."

If educational statisticians would only use the word "consistency" for what they call "reliability," they could then use "reliability" in its long established sense, without having to adopt a less familiar word, "validity."

A third point to which a mathematician might take exception concerns the "correction for attenuation"—i.e., the formula by which it is claimed that one may infer what the coefficient of correlation between two tests would be if there were no errors of measurement. Until the reviewer learns of a careful study of the probable error of such a corrected coefficient for small samples, he will feel that

the correction can be used only with much more caution than Dr. Otis suggests. Most students of statistics, probability, and philosophy regard the determination of the real that underlies the stream of appearance as a task calling for prayer and fasting. Yet this magic formula professes to remove the effect of errors of measurement on the calculation of one of the most tricky inventions of the statistician—and to do it reliably even for rather small samples.

In spite of a few points such as these mentioned, Dr. Otis's text seems to the reviewer acceptable to the mathematician or mathematical student who is making his first contact with educational statistical methods. Indeed, it may be recommended as a sort of antidote to a point of view too purely mathematical—as a glimpse of a world where ease in application is a more important characteristic of a proposed method than rigorous accuracy, and where informal discussions of a new principle that link it with accepted notions are valued more highly than logical deduction from definitely formulated postulates.

R. W. BURGESS

A Course of Geometrical Analysis. By HARIDAS BAGCHI. Calcutta, Chuckervertty, Chatterji and Co., 1926. 8+562 pages..

The author, Haridas Bagchi, M.A., Ph. D., Premchand Roychand Scholar and Lecturer in Pure Mathematics, Calcutta University, here makes an "attempt to harmonize as far as practicable Geometry and Analysis." This statement is of course further explained. A scrutiny of the book shows at once that the subject matter can be described as conventional differential geometry with an occasional timely excursus into some other mathematical field. Numerous references are made to standard British texts in algebra, calculus, theory of functions, vector analysis, and differential geometry. A few references are made also to Eisenhart's work on differential geometry and to some continental writers, e.g. "Wierstrauss" and "Lami."

The book is unusual and interesting. It shows clearly an inquisitive mind not content to lie bound among formulas and theorems. Apparently (although internal evidence may be misleading) the author has struggled through—and mastered—largely through his own efforts, Forsyth's Differential Geometry, keeping his mind open to possible applications, and has met with frequent difficulties. Some of these were cleared up by his own further reading of this same excellent text, but the pursuit of others has led him far afield. Attempting to teach the subject to others who may have shown less enthusiasm and less natural inquisitiveness has convinced the author that an isolated hierarchy of formal symbols may prove almost meaningless, while the fault lies not in the subject but in the exposition. This text taken by itself is hardly satisfying as a fundamental basis of instruction. Its many references to standard works will not prove too numerous to a critical reader who is accustomed to formal

proofs. But its detailed explanations, its suggestive and original comments, its interesting applications, its frequent cautions, and its relative freedom from specialized symbols and technical notations, make it a delightful contrast to most treatises on differential geometry and a particularly serviceable aid in self instruction. Perhaps a more severe author would have relegated much of the discussion to exercises, by relying upon experienced class room instruction to cover the technical difficulties. The book has no index, and the table of contents, even aided by the page of summary preceding each chapter, is not sufficient to make easy reference practicable. One misses the flavor of Euclidean precision in the details of proofs and this casts some doubt upon the validity of many of the statements, except where more ample proofs are found in the references. There appears to be not a little new material, and at least much here found is not readily accessible in other standard texts. Occasional remarks reveal however sad gaps in the author's scholarship, as might be expected in a work touching upon so many fields. Without intending any restrictions upon the functions employed, the author refers as follows to the solutions of Cauchy's familiar functional equations: "If a function conforms to the Law of Addition it must conform also to that of Multiplication, and vice versa. In either case, the function must have need to be linear and homogeneous. Further functions of this stamp constitute the only class to be thought of in connection with the afore-mentioned Laws." His proof uses differentiation. In an alternative arithmetic "proof," that $f(u+v)=f(u)+f(v)$, implies, $f(uv)=vf(u)$, he remarks after discussing the case of v rational, "There is no denying the fact that, while the Multiplicative Law, taken by itself, counts as a handy functional equation, its deduction from the Law of Addition is by no means an easy one, when v happens to be a transcendental, or even an incommensurable number. In all likelihood, modern speculations on the Theory of Numbers may lead to a simple solution of the thorny problem referred to."

The diction is frequently quaint although rarely ungrammatical, and is certainly excellent for one whose native tongue is not even European. One may be excused a goodnatured smile at reading, "The needment in the next stage can easily be met with by the students themselves, who should put the last hand to the problem," or "The constants a and b are as arbitrary as anything can be," or again "Where x and y behave like parameters, and as such flatly refuse to be struck out by elimination."

ALBERT A. BENNETT

ARTICLES IN CURRENT PERIODICALS

The lists appearing regularly in the Monthly of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

Annals of Mathematics, volume 28, no. 2, April 1927: "Some relations between a continuous curve and its subsets" by H. M. Gehman, 103–111; "Transformations of one principal equation into another" by R. Garver, 112–116; "On related maxima and minima" by N. Miller, 117–126; "On Fredholm's integral equations, whose kernels are analytic in a parameter" by J. D. Tamarkin, 127–152; "On the existence of the absolute minimum in space problems of the calculus variations" by L. M. Graves, 153–170; "The mixed mean function" by C. A. Shook, 171–179; "The focal point for the problem of Lagrange with one variable end point" by L. H. McFarlan, 180–195; "Differential invariants of affinely connected manifolds" by T. Y. Thomas and F. D. Michal, 196–236; "Linear congruences in a general arithmetic" by H. L. Olson, 237–239; "Note on a differential equation" by J. M. Thomas, 240–244; "On the class of a covariant tensor" by J. W. Alexander, 245–250.

American Journal of Mathematics, volume 49, no. 2, April 1927: "A class of functions harmonic within the sphere" by G. C. Evans, 153–180; "Notes on formal modular protomorphs" by O. C. Hazlett, 181–188; "Irreducible continuous curves" by H. M. Gehman, 189–196; "The plane quintic with five cusps" by M. Lehr, 197–214; "The convergence of general means and the invariance of form of certain frequency functions" by E. L. Dodd, 215–220; "On the interpolatory properties of a linear combination of continuous functions" by D. V. Widder, 221–234; "Note on Tchebycheff approximation" by D. V. Widder, 235–240; "Generalizations of Waring's theorem on fourth, sixth, and eighth powers" by L. E. Dickson, 241–250; "On complete systems of irrational invariants of associated point sets" by C. M. Huber, 251–267; "Regular maps and their groups" by H. R. Brahana; "The groups belonging to a linear associative algebra" by A. Ranum, 285–308.

Bulletin of the American Mathematical Society, volume 33, no. 4, July-August 1927: "A class of transcendental numbers" by S. Mandelbrojt, 413–415; "Cauchy's cyclotomic function and functional powers" by E. T. Bell, 416–422; "A point set which has no true quasi-components, and which becomes connected upon the addition of a single point" by R. L. Wilder, 423–426; "A simple method for normalizing Tchebycheff polynomials and evaluating the elements of the allied continued fractions" by J. A. Shohat, 427–432; "On Hilbert's thirteenth Paris problem" by H. W. Raudenbush, Jr., 433–434; "Generalization of the Beltrami equations to curved n-space" by G. E. Raynor, 435–438; "The non existence of a certain type of regular point set" by R. L. Wilder, 439–446; "A theorem of Frobenius on quadratic forms" by P. Franklin, 447–452; "On the partitions of a group and the resulting classification" by J. W. Young, 453–460; "Analytic functions with assigned values" by P. Franklin, 461–466; "An application of analysis situs to statistics" by H. Hotelling, 467–476; "A generalization of recurrents" by M. Ward, 477–490.

Journal of the London Mathematical Society, volume 2, no. 6, April 1927: "The zeros of certain integral functions" by J. D. Tamarkin, 66–69; "On a theorem of Bochner and Hardy" by N. Wiener, 118–123.

Journal of Mathematics and Physics, Mass. Inst. of Technology, volume 6, no. 4, June 1927: "A geometric characterization of equipotential and stream lines" by P. Franklin, 191–208; "On the conditions of validity of macromechanics" by M. S. Vallanta, 209–222; "Contact transformations in intrinsic geometry" by S. D. Zeldin, 223–239; "Convergent interpolation coefficients with convergent sum" by G. Rutledge, 240–249.

Mathematische Annalen, volume 97, nos. 4–5, June 1927: "On the reality of singularities of plane curves" by T. R. Hollcroft, 775–787.

Mathematische Zeitschrift, volume 26, no. 5, June 1927: "Concerning the sum of a countable infinity of mutually exclusive continua" by J. R. Kline, 687–690.

Proceedings of the National Academy of Sciences, U. S. A., volume 13, no. 4, April 1927: "On the expansion of harmonic functions in terms of harmonic polynomials" by J. L. Walsh, 175–179. No. 5, May 1927: "Concerning points on a continuous curve which are not accessible from each other" by C. M. Cleveland, 275; "Recent progress of investigations by symbolical methods of the invariants of biternary quantics" by O. E. Glenn, 276–279; "A dynamical theory of economic equilibrium" by C. F. Roos, 280–285; "On the foundations of the theory of discontinuous groups of linear transformations by combination" by L. R. Ford, 289–290. No. 6, June 1927: "Groups of collineations in a space of paths" by M. S. Knebelman, 396–399.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3280. Proposed by Philip Fitch, Denver Public Schools.

If a flexible chain, suspended at the ends from points in a horizontal line, is loaded so that the load varies directly as the square of the distance along the horizontal from the mid-point of the chain, find the equation of the curve made by the chain.

3281. Proposed by Emma M. Gibson, High School, Springfield, Mo.

Two particles of masses m_1 and m_2 are tied by fine inextensible strings to a third particle of mass m_3. They lie on a rough plane, whose inclination to the horizon (θ) is less than the angle of friction (ϵ), with the strings stretched and making angles α and β with the line of intersection of the plane with the horizontal plane. Find the magnitude and direction of the least horizontal force which, on being applied to the third particle, will move all three.

3282. Proposed by H. Halperin, A. & M. College of Texas, College Station, Texas.

Justify the following method of solving an exact differential equation of the first order, $Mdx+Ndy=0$: Replace y in M and N by ux and dy by udx; integrate the resulting expression with regard to x, considering u as constant; replace u by y/x, equate to a constant, C, and obtain the solution.

3283. Proposed by W. L. Ayres, University of Pennsylvania.

Let C_1 and C_2 be the circles $x^2+y^2=1$ and $x^2+(y-b)^2=r^2$, where $r+1>b>r>0$. Let L_1 be the upper arc of C_1 and L_2 the lower arc of C_2 between the points of intersection of the two circles. If the volume of the circular ring formed by revolving L_1+L_2 about the x-axis is a constant V, find b and r such that the area of the portion of the surface of the ring generated by L_2 may be a minimum.

3284. Proposed by Burrell Morgan, Krollitz, W. Va. and Norman Anning, University of Michigan.

In a given sphere is inscribed the maximum right prism whose bases are regular polygons of a given number of sides. Show that the altitude of the prism is independent of the number of its lateral faces.

3285. Proposed by R. H. Sciobereti, University of California.

Find the most general function $f(x)$ such that the integral $\int_a^b \phi(x)\, f(x-h)dx$ should have a value independent of h, where $\phi(x)$ is a given continuous function. Examine the cases where one of the limits or both become infinite.

3286. Proposed by V. M. Spunar, Chicago, Illinois.

A doubly infinite system of similar conics in parallel planes have their centers collinear and their corresponding axes parallel. Show that they can be cut orthogonally by a family of surfaces only if the line of centers is perpendicular to their planes.

3287. Proposed by D. M. Yost, California Institute of Technology.

Evaluate the definite integral $\int_0^\infty (x^n/x^x)dx$.

SOLUTIONS

3176[3172; 1926, 105]. Proposed by J. B. Reynolds, Lehigh University.

A linkage consisting of four equal uniform rods each of length $2a$ and weight w, loosely jointed in the form of a rhombus in a vertical plane, carries a weight P at the lowest vertex and is supported by the two upper rods resting against a smooth horizontal cylinder of radius r; find the time of a small vibration of the system in a vertical plane.

SOLUTION BY THE PROPOSER.

Assume the x-axis horizontal in the plane of motion, the y-axis vertically upward, and the origin at the center of the cylinder. Let (x, y) be the coordinates of the center of the right upper rod, (x', y') of the center of the right lower rod, and (X, Y) of the weight P. Suppose that in equilibrium the radius of the cylinder to the point of contact of the upper rod makes an angle A with the vertical. Let the system be slightly displaced in a vertical plane so that the radius to the point of contact makes an angle $A-\theta$ with the vertical. Then we shall have

$$x = a\cos(A-\theta) \qquad y = r\sec(A-\theta) - a\sin(A-\theta)$$
$$x' = x \qquad y' = y - 2a\sin(A-\theta)$$
$$X = 0 \qquad Y = y' - a\sin(A-\theta).$$

Let K be the kinetic energy of the oscillating system and Q its potential energy with reference to the position of equilibrium. We shall then have

$$gK = w(\dot{x}^2 + \dot{y}^2 + \dot{x}'^2 + \dot{y}'^2) + (P/2)(\dot{X}^2 + \dot{Y}^2) + (2w/3)a^2\dot{\theta}^2$$

in which the dots indicate derivatives with respect to time; and

$$Q = 2w(y - y_0) + 2w(y' - y_0') + P(Y - Y_0)$$

in which the subscripts refer to the values of the coordinates in the position of equilibrium.

Expanding to the squares of small terms in θ we find

(1) $$gK = \{(1/2)(4w + P)r^2\sec^2 A\tan^2 A - 4(2w + P)ar\tan A + 2[(4/3)w + 4(w + P)\cos^2 A]a^2\}\dot{\theta}^2$$

and

(2) $$Q = [(8w + 4P)a\cos A - (4w + P)r\sec A\tan A]\theta + (1/2)[(4w + P)r\sec A(2\sec^2 A - 1) + (8w + 4P)a\sin A]\theta^2.$$

Since Q is a minimum for $\theta = 0$ we must have

(3) $$(4w + P)r\sin A = 4(2w + P)a\cos^3 A,$$

which determines A for the position of equilibrium. This result could have been obtained from the equations of equilibrium for the system.

By means of (3) we may now write (1) and (2) in the forms

$$gK = (4aw/3)(2a - 6a\cos^2 A + 3r\tan A)\dot{\theta}^2 = H\dot{\theta}^2,$$

and

$$Q = 2a(2w + P)(2\sin A + \operatorname{cosec} A)\,\theta^2 = G\theta^2.$$

Now $K+Q=(H/g)\dot{\theta}^2+G\theta^2=$constant and, upon differentiating, we find $H\ddot{\theta}+gG\theta=0$ whence, if T is the time of a complete oscillation,

$$T = 2\pi(H/Gg)^{1/2} = (2\pi/3)[6w\sin A(2a - 6a\cos^2 A + 3r\tan A)/(2w + P)(2\sin^2 A + 1)g]^{1/2},$$

in which A is in the first quadrant and is determined by equation (3).

Also solved by WILLIAM HOOVER.

3217[1926; 480]. Proposed by R. H. Sciobereti, Berkeley, Calif.

Find a curve such that the radius of curvature at any point is proportional to the reciprocal of the normal.

SOLUTION BY EUGENE M. BERRY, Lynchburg College, Lynchburg, Va.

Putting the curvature equal to a times the normal, we have the differential equation

$$(1) \qquad (d^2y/dx^2)/[1 + (dy/dx)^2]^{3/2} = ay[1 + (dy/dx)^2]^{1/2}.$$

If we put $p = dy/dx$ and $p(dp/dy) = dy^2/dx^2$ this becomes $pdp/(1+p^2)^2 = aydy$.

If we integrate and solve for p, we get $p = \pm[(1-c+ay^2)/(c-ay^2)]^{1/2}$, whence

$$(2) \qquad x = \pm \int [(c - ay^2)/(1 - c + ay^2)]^{1/2} dy.$$

If $c=1$ in equation (2), we have

$$(3) \qquad x = \pm \frac{1}{\sqrt{a}} \left\{ \sqrt{(1 - ay^2)} - \log\left(\frac{1 + \sqrt{(1 - ay^2)}}{y\sqrt{a}} \right) \right\} + C$$

which is the equation of a tractrix.

If $c=0$ in equation (2), we have $x = \pm[(1+ay^2)/(-a)]^{1/2} + C$, which is the equation of a circle.

With these two exceptions the integrals in the above equations are elliptic integrals. If the curves are to be real, we must have either $a>0$, $c>0$ or $a<0$, $c<1$. We can put these in type forms as follows.

When $a>0$, $0<c<1$, put $y = (c/a)^{1/2} \cos\theta$; equation (2) then becomes

$$(5) \qquad x = \pm a^{-1/2} \int [(1 - c \sin^2\theta)^{1/2} - (1 - c \sin^2\theta)^{-1/2}] d\theta.$$

When $a>0$, $c>1$, put $y = [(c - \sin^2\theta)/a]^{1/2}$; equation (2) then becomes

$$(6) \qquad x = \pm (c/a)^{1/2} \int [(1 - c^{-1} \sin^2\theta)^{1/2} - (1 - c^{-1} \sin^2\theta)^{-1/2}] d\theta.$$

When $a<0$, $0<c<1$, put $y = [(1-c)/(-a)]^{1/2} \cos\theta$; equation (2) becomes

$$(7) \qquad x = \mp(-a)^{-1/2} \int [1 - (1 - c) \sin^2\theta]^{1/2} d\theta.$$

When $a<0$, $c<0$, put $y = [(\cos^2\theta - c)/(-a)]^{1/2}$; equation (2) becomes

$$(8) \qquad x = \mp(-a)^{-1/2} \int [(1-c)^{1/2} \{1 - (1-c)^{-1} \sin^2\theta\}^{1/2} + c(1-c)^{-1/2} \{1 - (1-c)^{-1} \sin^2\theta\}^{-1/2}] d\theta.$$

Also solved by J. B. REYNOLDS and the PROPOSER.

3219[1926, 480]. Proposed by Philip Fitch, Denver, Colorado.

Construct a polygon similar to a given polygon and having the reciprocal of its area equal to the sum of the reciprocals of the areas of a certain number of given polygons.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

To find graphically a line of length H such that for two given lines of lengths h_1 and h_2 there shall exist the relation $1/H = 1/h_1 + 1/h_2$, draw rays OA, OB, OC, and OD in a plane making successive angles of 60° each. On OA lay off $OA = h_1$, on OC lay off $OC = h_2$. Let AC cut OB at B. Let $OB = H$. Then Area AOC = Area AOB + Area BOC, or $h_1h_2 \sin 60° = h_1H \sin 60° + h_2H \sin 60°$; and therefore $1/H = 1/h_1 + 1/h_2$.

On OD lay off $OD = h_3$. Draw BD cutting OC at E. Let $OE = H'$.

Then $1/H' = 1/h_3 + 1/H = 1/h_1 + 1/h_2 + 1/h_3$.

This process can be continued to any number of reciprocals. Now let there be n given polygons of areas $A_1, A_2, \cdots, A_n$; to find A such that $1/A = 1/A_1 + 1/A_2 + \cdots + 1/A_n$.

Construct n right triangles each of unit base and altitudes $h_1, h_2, \cdots, h_n$ whose areas are equal to $A_1, A_2, \cdots, A_n$, respectively. Find h by the relation $1/h = 1/h_1 + 1/h_2 + \cdots + 1/h_n$. Construct a right triangle of unit base and altitude h; then $h = 2A$, $h_1 = 2A_1$, $h_2 = 2A_2, \cdots, h_n = 2A_n$; whence $1/A = 1/A_1 + 1/A_2 + \cdots + 1/A_n$, giving A, the area of the required polygon.

Construct two isosceles right triangles of areas A and A' and legs a and a', where A' is the area of the given polygon. Construct the required similar polygon with its sides in the ratio a/a' to those of the given polygon.

NOTE BY THE EDITORS

One could say: Construct n rectangles each of unit base with altitudes $h_1, h_2, \cdots$, whose areas are $A_1, A_2, \cdots$. Then $h_1 = A_1$, $h_2 = A_2, \cdots$.

Also solved by MICHAEL GOLDBERG and the PROPOSER.

3221[1926, 480]. Proposed by H. E. Trefethen, Colby College.

A variable rectangle has a diagonal of constant length and two sides lying upon two fixed perpendicular straight lines. Determine geometrically the locus of the foot of the perpendicular from the vertex opposite the fixed vertex upon the diagonal which does not pass through that vertex.

SOLUTION BY ROSCOE WOODS, University of Iowa.

Let the vertices of the variable rectangle be O, A, B, C, where O is the fixed vertex, A and C are vertices on the two fixed perpendicular straight lines, and B the remaining vertex whose locus is evidently a circle, K, whose center is O and whose radius is r which is the length of the constant diagonal of this variable rectangle. Consider the rectangle in one of its possible positions and draw the diagonals OB and AC and let M be their point of intersection. From B draw a perpendicular to AC, cutting AC in P. The triangle MPB is a right triangle so that a circle K' on BM as diameter always passes through P. Since BM is a constant the radius of K' is one-fourth the radius OB of K. Since BM, the diameter of K', lies along the radius OB of K, K and K' are tangent at B. Let S be the center of K'. From a figure, it is easily seen that the angle PSB is four times the angle AOB, which shows that the arc subtending the angle PSB in K' is equal in length to the arc subtending the angle AOB in K. This proves that P is a fixed point of the circumference of K' if it rolls on K; therefore the locus of P is a hypocycloid of four cusps.

Also solved by J. M. BARBOUR, THEODORE BENNETT, ALICE M. GRANT, MICHAEL GOLDBERG, J. B. REYNOLDS, and the PROPOSER.

3223[1926, 481]. Proposed by Paul Capron, U. S. Naval Academy.

A circle of radius b and a straight line at a distance a from the center of the circle, lie in the same plane; the circle is revolved about the line generating a torus. A plane Π is passed through the axis l, intersecting the torus in two circles S_1 and S_2; a plane Σ is passed perpendicular to Π and containing the common interior tangent of S_1 and S_2. Show that Σ intersects the torus in two circles.

SOLUTION BY R. H. SCIOBERETI, University of California.

Let Oz be the axis of the torus; OX and Ox, the traces of the bitangent plane Σ and of any meridian on the equator, respectively. Let us denote by M one of the points of intersection of the plane Σ with the meridian circle S' of center C so that $OC = a$, $CM = b$. Let P and Q be the orthogonal projections of M on Ox and OX respectively and R the projection of point O on CM. Then we shall have angle $PQM = \theta$, where θ denotes the angle between the bitangent and the equatorial planes; hence,

$$\sin PQM = \sin\theta = b/a = PM/QM = CM/OC. \tag{1}$$

On the other hand the area of the triangle OCM evaluated in two different ways gives

$$OC \cdot PM = CM \cdot OR, \qquad \text{or} \qquad CM/OC = PM/OR,$$

and from a comparison with (1) it follows that $QM = OR$; hence $MR = OQ$, since the two right triangles MRO and OQM are congruent.

Let us now consider a point D on OX at a distance b from O, and such that the three points O, Q, D will be in the same order as the points M, R, C, so that $QD = OD - OQ = CM - MR = RC$; hence, the two right triangles DQM and CRO are congruent and consequently $DM = CO = a$. When the meridian circle

S' is rotated about Oz, point M will describe a circle of center D and of radius a. A similar reasoning shows that the locus of the second point of intersection M' of the plane Σ with the meridian circle S' is a circle equal to the first one, but whose center is the symmetrical point of D with respect to O.

REMARK I. These two circles, known as the Villarceau circles, intersect the parallels of the torus at a constant angle. This property may be shown by means of elementary geometry as follows: Let MT be the tangent to the Villarceau circle passing through M, and let MV be the tangent to the parallel through M; MT lies in the plane ODM and is orthogonal to DM, whereas MV is perpendicular to the plane OCM. Since the skew quadrilateral $ODMC$ has its opposite sides equal to each other, it follows that the two triangles ODM and MCO are congruent; hence $DH=CH$, where H denotes the point bisecting OM. In a similar manner it may be shown that if K is the mid-point of DC, $OK=MK$. Hence the line HK is the common perpendicular to OM and DC at their mid-points. This line HK may, therefore, be considered as an axis of symmetry for the skew quadrilateral. Now the symmetrical line of the tangent MT is the perpendicular to CO at the point O in the plane COM; hence it is Oz. The symmetrical line of the tangent MV is a line OU through O orthogonal to the plane DOM; then since angle VMT=angle $UOz=\theta$, it follows that the Villarceau circles intersect the parallels of the torus at a constant angle θ.

REMARK II. The projection of the Villarceau circle on the equator is an ellipse with one focus at point O. In fact its center is at D and its major axis, which is along OX, is $2a$; the minor axis is $2a \cos \theta$; hence, the focal distance is $a \sin \theta = b$ which proves the proposition.

The solver sent in also an analytical solution.

Also solved by THEODORE BENNETT, PAUL CAPRON, RUFUS CRANE, MICHAEL GOLDBERG, HARRY LANGMAN, and J. B. REYNOLDS.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kuhn, Ohio State University, Columbus, Ohio.

The second Madison Colloquium of the American Mathematical Society was held during the week of September 5, in connection with the meetings of the Society and the Association. The lecturers were Professor E. T. Bell of the California Institute of Technology and Professor Anna Pell Wheeler of Bryn Mawr College. They gave five lectures each, the former on "Algebraic Arithmetic" and the latter on "The Theory of Quadratic Forms in Infinitely many Variables and Applications."

The long delay in publishing the Rhind Mathematical Papyrus is due to unforeseen difficulties in connection with the correction of the 140 plates by the hieroglyphic expert. It is now confidently expected that the two volumes will be ready for delivery in December. Personal explanation will be made by letter to all advance subscribers already filed. Further subscriptions will be filed in the order of their receipt. In the November Monthly there will appear a complete formal description of this great work on the eve of its appearance. The description will be written by Professor R. C. Archibald.

A $1000 fellowship in mathematics has recently been established at Brown University by Mr. H. D. SHARPE.

At the Armour Institute of Technology, Professor DONALD F. CAMPBELL has resigned; and Associate Professor C. I. PALMER has been promoted to a full professorship of mathematics. Professor Palmer is also acting dean of students.

Mr. HERBERT E. ARNOLD has been promoted to an assistant professorship of mathematics at Wesleyan University.

Assistant Professor L. C. BAGBY of the University of South Dakota has been appointed professor of mathematics at the Linsly Institute of Technology.

Mr. M. A. BASOCO, who was reported last month as appointed to an instructorship at the University of California, Los Angeles, has accepted a graduate fellowship at the California Institute of Technology.

Mr. A. H. BLUE of the University of Iowa has been elected professor of mathematics at West Union College.

Assistant Professor EVELYN T. CARROLL of Wells College has been granted leave of absence for the year 1927–28.

Mr. T. F. COPE, instructor last year at Western Reserve University, has been appointed instructor in mathematics at the University of Chicago for 1927–28.

Professor L. S. DEDERICK of St. Stephen's College has been appointed mathematician at the Aberdeen Proving Ground.

Associate Professor H. J. ETTLINGER of the University of Texas has been promoted to a full professorship of mathematics.

Assistant Professor M. C. FOSTER of Williams College has been appointed associate professor of mathematics at Wesleyan University.

Mr. D. E. HARKIN, who has just received the doctorate at the University of Chicago, has been appointed professor of mathematics at Alabama Polytechnic Institute.

Miss MARIE JOHNSON, who was reported last month as appointed to an instructorship at Pennsylvania State College, has been released to accept an assistant professorship at Oberlin College.

Assistant Professor C. G. LATIMER of Tulane University has been appointed professor of mathematics at the University of Kentucky.

Professor P. H. LINEHAN of the College of the City of New York has been made director of the evening session and of the division of vocational studies.

Mr. W. T. MacCreadie of Cornell University has been appointed assistant professor of mathematics at Bucknell University.

Mr. T. H. Milne has been appointed assistant professor of mathematics at the University of Manitoba.

Miss Ethel I. Moody, M.A., Cornell, 1927, has been appointed instructor in mathematics at Wells College.

Dr. J. H. Neelley of Yale University has been appointed associate professor of mathematics at the Carnegie Institute of Technology.

Mr. J. C. Nixon, S.M., University of Chicago, March, 1927, has accepted an instructorship at the University of Alabama.

Associate Professor R. S. Underwood of Alabama Polytechnic Institute has been appointed associate professor of mathematics at Texas Technological College.

Assistant Professor E. E. Whitford of the College of the City of New York has been promoted to an associate professorship of mathematics.

Dr. W. H. Wilson of the University of Iowa has been appointed associate professor of mathematics at the University of Florida.

Dr. H. A. Zinszer of the Mississippi State College for Women has been appointed professor of mathematics at Hanover College.

The following appointments to instructorships in mathematics are announced:

Cornell University, Mr. C. C. Torrance;
University of Florida, Mr. C. A. Messick of the University of Iowa;
Rutgers College, Mr. C. R. Wilson of the University of Iowa;
Northwestern University, Mr. H. L. Garabedian;
Yale University, Mr. T. C. Benton.
Junior College at Jefferson City, Miss Frances Baker of Tabor College.

Professor W. G. Bullard of Syracuse University died February 16, 1927.

Professor W. W. Johnson of the United States Naval Academy died May 14, 1927, at the age of eighty-three.

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The Association needs funds for scientific publications and for the promotion of scientific activities.

Two such funds have already been started:
THE CARUS PUBLICATION FUND and
THE ARNOLD BUFFUM CHACE FUND.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to W. B. CARVER, White Hall, Ithaca, N. Y.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eleventh Summer Meeting of the Association, Madison, Wisconsin, September 5-6, 1927.

Twelfth Annual Meeting, Nashville, Tenn., December, 1927.

The following are dates of Section Meetings of the Association in 1927:

ILLINOIS, Bloomington, Ill., May 13-14.

INDIANA, De Pauw University, April 29-30.

IOWA, University of Iowa, May 6-7.

KANSAS, Topeka, Kan., February 5.

KENTUCKY, Lexington, May 14.

LOUISIANA-MISSISSIPPI, Shreveport, La., March 4-5.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May 7 and Georgetown University, December 3.

MICHIGAN, April.

MINNESOTA, St. Peter, Minn., May 21.

MISSOURI, St. Louis, Mo., November 25-26.

NEBRASKA, Lincoln, May 6.

OHIO, Columbus, Ohio, April 8.

PHILADELPHIA, Philadelphia, Pa., November 26.

ROCKY MOUNTAIN, Colorado College, April 22-23.

SOUTHEASTERN, Columbia, S. C., April 15-16.

SOUTHERN CALIFORNIA, Los Angeles, Calif., March 12 and November 5.

TEXAS, Not yet determined.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE LOUISIANA-MISSISSIPPI BRANCH OF THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE RHIND MATHEMATICAL PAPYRUS

The following description of the forthcoming publication of the oldest known mathematical document in the world will be of interest to readers of the Monthly.

British Museum 10057 and 10058. Photographic Facsimile of the Rhind Mathematical Papyrus, Hieroglyphic Transcription, Transliteration, Literal Translation, Free Translation, Mathematical Commentary and Bibliography. In two sumptuous volumes.

Volume I, $11\frac{1}{4}\times 8$ inches: Free Translation and Commentary by ARNOLD BUFFUM CHACE, Chancellor of Brown University, with the assistance of HENRY PARKER MANNING: Bibliography of Egyptian Mathematics by RAYMOND CLARE ARCHIBALD. 8+210 pp.

Volume II, $11\frac{1}{4}\times 14\frac{1}{4}$ inches: Photographic Facsimile, Hieroglyphic Transcription, Transliteration, Literal Translation, by ARNOLD BUFFUM CHACE, with the assistance of LUDLOW BULL, Metropolitan Museum of Fine Arts, New York, and HENRY PARKER MANNING; also Leather Roll, British Museum 10250, facsimile with description by S. R. K. GLANVILLE, British Museum; in all 140 plates with text and introductions, all the plates and text of the Rhind Papyrus being in the two colors (red and black) of the original.

To be published by the Mathematical Association of America, Oberlin, Ohio, U. S. A., in December 1927. Price to individual and institutional members of the Association for personal use, $15.00 for the set. To all others $20.00.

This forthcoming work on the Rhind Papyrus is a publication of great interest and value not only to the student of mathematics but also to almost anyone interested in a civilization of nearly four thousand years ago. Through the courtesy of the British Museum authorities, Chancellor Chace is able to give for the first time a complete photographic reproduction of this very important mathematical work written about 1650 B.C. in the hieratic language, but copied from a work considerably older. Other plates give, under the hieratic writing, which is from right to left, the corresponding hieroglyphic transcription and transliteration, while on the opposite page is the transliteration from left to right and the literal translation. The first volume contains the free translation with detailed commentary and Professor Archibald's elaborate bibliography of Egyptian mathematics which, with its indexes, occupies 86 pages.

It was not till the summer of 1927 that a distinguished English chemist succeeded in unrolling another mathematical document in the British Museum,